Long time existence for the semi-linear Klein-Gordon equation on a compact boundaryless Riemannian manifold

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LONG TIME EXISTENCE FOR THE SEMI-LINEAR KLEIN-GORDON EQUATION ON A COMPACT BOUNDARYLESS RIEMANNIAN MANIFOLD

by

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Abstract. — We investigate the long time existence of small and smooth solutions for the semi-linear Klein-Gordon equation on a compact boundaryless Riemannian manifold. Without any spectral or geometric assumption, our first result improves the lifespan obtained by the local theory. The previous result is proved under a generic condition of the mass. As a byproduct of the method, we examine the particular case where the manifold is a multidimensional torus and we give explicit examples of algebraic masses for which we can improve the local existence time. The analytic part of the proof relies on multilinear estimates of eigenfunctions and estimates of small divisors proved by Delort and Szeftel. The algebraic part of the proof relies on a multilinear version of the Roth theorem proved by Schmidt.

Contents

1. Introduction

Let us consider a smooth compact boundaryless Riemannian manifold X (with $d := \dim X \geq 2$) and denote by Δ its negative Laplace-Beltrami operator. We investigate the dynamics of the solutions of the Klein-Gordon equation

$$(\partial_t^2 - \Delta + m^2)w = w^{n+1}, \quad x \in X, \quad t \in \mathbb{R}. \tag{1}$$

In the previous equation, n is a positive integer and m is a positive real number (usually called the mass). In this work, we are concerned with very smooth solutions (namely which belong to $H^s(X)$ with $s \gg 1$). A fix point argument easily leads to the following result : if $\varepsilon \in (0,1)$ is small enough and if $(w(0,\cdot), \dot{w}(0,\cdot)) \in H^s(X) \times H^{s-1}(X)$ is an initial datum such that $\|w(0,\cdot)\|_{H^s(X)} + \|\dot{w}(0,\cdot)\|_{H^{s-1}(X)}$ is less than ε, then the Klein-Gordon equation (1) admits a unique solution which is bounded by $C\varepsilon$ on a lifespan of order ε^{-n}. Our purpose is to extend the lifespan given by the local theory and to keep the uniform control of the solution by $C\varepsilon$. For simplicity, we call ε^{-n} the local existence time.

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Such a question was studied in several articles and may be summarized as follows : is there a constant $A > 1$ such that we can improve the local existence time to ε^{-An} ?

One can organize the known results in two categories depending on the asymptotic behavior of the spectrum of $\sqrt{-\Delta}$. The first case that had been understood is the one-dimensional torus $\mathbb{T}$ (see [**Bou96, Bam03, BG06**]). Provided that $m > 0$ is chosen outside a zero Lebesgue measure subset of $(0, +\infty)$, the Klein-Gordon equation admits *almost global solutions* on $\mathbb{T}$: the local existence time can be improved to $C_{A,n}\varepsilon^{-An}$ for any real number $A > 1$. Such a result has been extended to multidimensional spheres $\mathbb{S}^d$ (or more generally Zoll manifolds) by Bambusi, Delort, Grébert, Szeftel (see [**BDGS07**]). In these cases, an important tool in the proofs is the separation of the spectrum of $\sqrt{-\Delta}$: roughly speaking, two different eigenvalues $\lambda \neq \lambda'$ of $\sqrt{-\Delta}$ fulfill a uniform estimate $|\lambda - \lambda'| \gtrsim 1$ (for Zoll manifolds, a weaker property holds). For the sequel of our work, it is worth pointing out that the previous paper [**BDGS07**] makes use of universal multilinear estimates of eigenfunctions (proved by Delort-Szeftel in [**DS06**]) that hold true without any geometric assumption (although merely used for Zoll manifolds).

The multidimensional torus $\mathbb{T}^d$ does not fulfill the above-mentioned property of separation. For instance, the spectrum of $\sqrt{-\Delta}$ on $\mathbb{T}^d$, with $d \geq 4$, is nothing else than the set $\{\sqrt{k}, k \in \mathbb{N}\}$. Two different and positive eigenvalues $\lambda \neq \lambda'$ of $\sqrt{-\Delta}$ consequently satisfy the inequality $|\lambda - \lambda'| \gtrsim (\lambda + \lambda')^{-1}$. However, the first author showed in [**Del09**] that the local existence time ε^{-n} can be improved to ε^{-An} with $A = 1 + \frac{2}{d}$ (up to a multiplicative logarithmic term). The previous result has been extended by Fang-Zhang in [**FZ10**] and Zhang also applied this new method in [**Zha10**] to a Klein-Gordon equation posed on $\mathbb{R}^d$ (with a quadratic potential which allows for a pure point spectrum). The second author remarked in [**Ime15**] that the paper [**Zha10**] has a counterpart on any compact boundaryless Riemannian manifold for which all eigenvalues of Δ belong to $\mathbb{Z}$ (for instance a finite product of spheres or Lie groups).

All the previous works use in an essential way a property of separation and suggest that the more separated the spectrum of $\sqrt{-\Delta}$ is, the better the improvement of the local existence time can be. Moreover, almost all results are obtained for almost every mass $m > 0$ (and such masses are thus not explicit). The only exceptions we know are the Klein-Gordon equation with a quadratic nonlinearity, i.e. $n = 1$, (see [**DS04**, Theorem 2.1] and [**Del98**] on tori or [**Zha16**] on $\mathbb{R}^d$ with the harmonic oscillator). The purpose of our article is to study in a unified approach the following two questions :

i) Can we improve the local existence time on any compact boundaryless Riemannian manifold ? A positive answer had been predicted in [**Del09**, page 165] if n is odd or if any two eigenvalues $\lambda \neq \lambda'$ of $\sqrt{-\Delta}$ fulfill an inequality of the form $|\lambda - \lambda'| \gtrsim |\lambda + \lambda'|^{-\beta}$ for some $\beta > 0$. We give below an affirmative answer without any assumption of parity on n and without separating all eigenvalues. Quite surprisingly, the main idea is that a weaker property of separation holds true whatever the manifold is. More precisely, if one denotes by $(\mu_j)_{j\in\mathbb{N}}$ the increasing sequence of all eigenvalues (without counting multiplicities) of $\sqrt{-\Delta}$, then Proposition 4 claims that there are a constant $C(X) > 0$ and a subsequence $(\mu_{j_k})_{k\in\mathbb{N}}$ such that the following holds as k tends to $+\infty$:

$$\mu_{j_k} = C(X)k + O(1) \quad \text{and} \quad \mu_{1+j_k} - \mu_{j_k} \gtrsim \frac{1}{\mu_{j_k}^{d-1}}. \tag{2}$$

This very weak property of separation allows to consider packets of eigenvalues as done in [**DS06, BDGS07**] for Zoll manifolds.

ii) For $X = \mathbb{T}^d$ with the usual metric and $n > 1$, can we give examples of *explicit* masses $m > 0$ for which we can improve the local existence time ? We give a very simple answer : one may choose any positive real algebraic number m whose degree is larger than 2^{n+1}, for instance $m = \sqrt[p]{3}$ is convenient for any integer $p > 2^{n+1}$.

The point to emphasize in our first result is that it excludes a blow-up of small solutions, whatever the geometry of X is, in a slightly longer time than the one given by the local theory.

Theorem 1. — *There exists a zero Lebesgue measure subset $\mathcal{E}_{n,X} \subset (0, +\infty)$ such that the following holds true for any $m \in (0, +\infty) \backslash \mathcal{E}_{n,X}$. There are two positive numbers $A > 1$ and $s_0 > 0$ (which only depend on (m, X, n)) such that for any $s > s_0$, for any couple of real-valued functions $(w_0, w_1) \in H^{s+1}(X) \times H^s(X)$ with $\|w_0\|_{H^{s+1}(X)} + \|w_1\|_{H^s(X)} = 1$, there are $C > 0$ and $K > 0$ such that if $\varepsilon > 0$ is small enough then the*

Klein-Gordon equation (1) *admits a unique solution*

$$w \in \mathcal{C}^0((-C\varepsilon^{-An}, +C\varepsilon^{-An}), H^{s+1}(X)) \cap \mathcal{C}^1((-C\varepsilon^{-An}, +C\varepsilon^{-An}), H^s(X)),$$

with initial data $(w(0), \dot{w}(0)) = (\varepsilon w_0, \varepsilon w_1)$. *Furthermore one has the uniform bound*

$$\forall t \in (-C\varepsilon^{-An}, +C\varepsilon^{-An}) \qquad \|w(t)\|_{H^{s+1}(X)} + \|\dot{w}(t)\|_{H^s(X)} \leq K\varepsilon.$$

We prove Theorem 1 with a normal form procedure which only relies on multilinear estimates of eigenfunctions and estimates of small divisors both obtained by Delort-Szeftel. In this dynamical context, the small divisors are ± 1-linear combinations of the eigenvalues of $\sqrt{-\Delta + m^2}$. Let us now recall the Delort-Szeftel estimates if n is odd : for almost every $m > 0$, there are $N_0 > 0$ and $C(m, n, N_0, d) > 0$ such that for all $(k_1, \ldots, k_{n+2}) \in \mathbb{N}^{n+2}$ and $(\omega_1, \ldots, \omega_n) \in \{-1, +1\}^{n+2}$ one has the lower bound

$$\left| \sum_{j=1}^{n+2} \omega_j \sqrt{\mu_{k_j}^2 + m^2} \right| \geq \frac{C(m, n, N_0, d)}{(1 + \max(\mu_{k_1}, \ldots, \mu_{k_{n+2}}))^{N_0}}. \tag{3}$$

The exponent $A > 1$ in Theorem 1 is directly linked to the exponent N_0 (see the formula (37)) and is therefore ineffective as it comes, among other arguments, from the Łojasiewicz inequality (see [**DS04**, Part 5]). By contrast, better versions of (3) are used in the papers [**Del09, FZ10, Zha10**] thanks to an adequate separation of *all eigenvalues*. This is why explicit constants $A > 1$ are obtained in the previous results. The most favorable case is that in which one can bound the left-hand side of (3) from below by a negative power of the third largest frequencies among $\mu_{k_1}, \ldots, \mu_{k_{n+2}}$ (for instance for spheres, see [**BDGS07**]).

Let us say a word on the difficulty if n is even. The left-hand side of (3) may become very small due to simultaneous compensations and it seems hopeless to consider separately each eigenvalue. Using (2) allows to overcome such an issue by considering separated packets of eigenvalues. The price to pay is to solve a multidimensional homological equation (see Proposition 10) and unfortunately have a small loss of a power of the largest frequency.

Let us now state our result about explicit masses on rational tori $\mathbb{T}^d$ and more generally on manifolds whose eigenvalues are integers (e.g. finite product of spheres). As far as we know, the following result is the first one that provides explicit examples of masses $m > 0$ for a nonlinearity which is at least cubic (the quadratic case is done in [**DS04, Del98**] for every $m > 0$ and in [**Zha16**] for $m \in 2\mathbb{N}$ in odd dimensions with a quadratic potential).

Theorem 2. — *Assume that all the eigenvalues of* Δ *are integers and that* m *is a positive real algebraic number of degree* $\deg(m) > 2^{n+1}$. *Consider moreover a real number* A *satisfying*

$$1 < A < 1 + \frac{2}{d - 1 + 2^{2n+2}}.$$

Then there exists $s_0 > 0$ *(which only depends on* (n, d, A)*) such that the following holds. For any* $s > s_0$, *for any couple of real-valued functions* $(w_0, w_1) \in H^{s+1}(X) \times H^s(X)$ *with* $\|w_0\|_{H^{s+1}(X)} + \|w_1\|_{H^s(X)} = 1$, *there are* $C > 0$ *and* $K > 0$ *such that if* $\varepsilon > 0$ *is small enough then the Klein-Gordon equation* (1) *admits a unique solution*

$$w \in \mathcal{C}^0((-C\varepsilon^{-An}, +C\varepsilon^{-An}), H^{s+1}(X)) \cap \mathcal{C}^1((-C\varepsilon^{-An}, +C\varepsilon^{-An}), H^s(X)),$$

with initial data $(w(0), \dot{w}(0)) = (\varepsilon w_0, \varepsilon w_1)$. *Furthermore one has the uniform bound*

$$\forall t \in (-C\varepsilon^{-An}, +C\varepsilon^{-An}) \qquad \|w(t)\|_{H^{s+1}(X)} + \|\dot{w}(t)\|_{H^s(X)} \leq K\varepsilon.$$

Note that the condition on A in Theorem 2 does not depend on m whereas it does in Theorem 1. To explain the algebraic assumption on m in the statement of Theorem 2, it is worthwhile to recall a few results in the theory of Diophantine approximations. The following result is very well-known and has a very elementary proof : almost every real number α, in the sense of Lebesgue, satisfies the following

$$\forall \delta > 0 \quad \exists C(\alpha, \delta) > 0 \quad \forall \frac{p}{q} \in \mathbb{Q} \quad \left| \alpha - \frac{p}{q} \right| \geq \frac{C(\alpha, \delta)}{|q|^{2+\delta}}. \tag{4}$$

One of the deepest theorems in the theory of Diophantine approximations states that the previous also holds true for any irrational algebraic number α (this is the Roth theorem, see [**Rot55**] or [**Bug04**, Theorem 2.1]). One can reformulate this result as follows : for any real algebraic number α

$$\forall \frac{p}{q} \in \mathbb{Q} \quad \left|\alpha - \frac{p}{q}\right| \neq 0 \quad \Rightarrow \quad \forall \delta > 0 \quad \exists C(\alpha, \delta) > 0 \quad \forall \frac{p}{q} \in \mathbb{Q} \quad \left|\alpha - \frac{p}{q}\right| \geq \frac{C(\alpha,\delta)}{|q|^{2+\delta}}. \tag{5}$$

It turns out that (4) has been extended by Sprindžuk (thus proving a conjecture made by Mahler, see [**Spr69**]) : for almost every $\alpha \in \mathbb{R}$ and every $\ell \in \mathbb{N}^\star$, we have

$$\forall \delta > 0 \quad \forall (q_0, \dots, q_\ell) \in \mathbb{Z}^{\ell+1} \backslash \{0\} \quad |q_0 + q_1\alpha + \cdots + q_\ell \alpha^\ell| \geq \frac{C(\alpha, \ell, \delta)}{\max(|q_0|, \dots, |q_\ell|)^{\ell+\delta}}. \tag{6}$$

Again, one has an algebraic counterpart : a consequence of a paper by Schmidt [**Sch70**] implies that (6) still holds true if α is a real algebraic number that satisfies $\deg(\alpha) > \ell$.

Let us go back to the Delort-Szeftel estimates (3) on the torus $\mathbb{T}^d$, so the eigenvalues of $-\Delta$ are nonnegative integers. The idea is simple to explain. Let us denote by $\mathbf{D}_\ell$ the set of the real number α satisfying (6). To avoid a discussion about resonant terms in this introduction, for any *odd* n we denote by $\mathbf{M}_n$ the set of the real numbers $\mu > 0$ for which there are $N_0 > 0$ and $C(\mu, n, N_0) > 0$ such that

$$\forall (k_1, \dots, k_{n+2}) \in \mathbb{N}^{n+2} \quad \forall (\omega_1, \dots, \omega_n) \in \{\pm 1\}^{n+2} \quad \left| \sum_{j=1}^{n+2} \omega_j \sqrt{\mu_{k_j}^2 + \mu} \right| \geq \frac{C(\mu, n, N_0)}{(1 + \max(\mu_{k_1}, \dots, \mu_{k_{n+2}}))^{N_0}}.$$

In the specific case where the inclusion $\mathrm{Sp}(-\Delta) \subset \mathbb{N}$ holds true, the proof of Proposition 24 will give us the following

$$\mathbf{D}_{2^n} \subset \mathbf{M}_n \subset \mathbb{R}.$$

The Sprindžuk theorem implies that $\mathbf{D}_{2^n}$ has full measure and so has $\mathbf{M}_n$. In other words, we get an alternative proof of a very particular case of the Delort-Szeftel estimates. We get a new result by using the Schmidt theorem : any real algebraic number μ satisfying $\deg(\mu) > 2^n$ belongs to $\mathbf{D}_{2^n}$ and so to $\mathbf{M}_n$. The natural relation $\mu = m^2$ finally explains the assumption $\deg(m) > 2^{n+1}$ of Theorem 2.

Analytic results	**Algebraic results**
Almost every $\alpha \in \mathbb{R}$ belongs to $\mathbf{D}_1$	Every real irrational algebraic number belongs to $\mathbf{D}_1$ (Roth)
$\uparrow$	$\uparrow$
For every $\ell \in \mathbb{N}^\star$, almost every $\alpha \in \mathbb{R}$ belongs to $\mathbf{D}_\ell$ (Sprindžuk)	For every $\ell \in \mathbb{N}^\star$, every real algebraic number of degree $> \ell$ belongs to $\mathbf{D}_\ell$ (Schmidt)
$\downarrow$	$\downarrow$
Almost every $\mu > 0$ belongs to $\mathbf{M}_n$ (Delort-Szeftel)	Every real algebraic number of degree $> 2^n$ belongs to $\mathbf{M}_n$ (our result)

Section 2 contains the proof of the weak property of separation (2) and the statements (analytic and algebraic) of the Delort-Szeftel estimates. Section 3 explains the general strategy of normal form (which fails in our issue). In Section 4, we give the proof of the loss of frequencies due to the multidimensional homological equation. In the next sections 5, 6 and 7, we explain the strategy of *partial normal form.* Section

8 is devoted to the Schmidt and Sprindžuk results about simultaneous Diophantine approximations. Section 9 contains the proof of a general result that implies, if n is odd, the inclusion $\mathbf{D}_{2^n} \subset \mathbf{M}_n$.

2. Eigenvalues, mass and small divisors

The spectrum of the operator $\sqrt{-\Delta}$ is pure point and we denote by $(\lambda_j)_{j\geq 0}$ the nondecreasing sequence of its eigenvalues (*with multiplicities*). The Weyl formula with remainder (see [**Hör68**] or [**SV97**, Theorem 1.2.1]) gives the following asymptotics

$$\forall \lambda \gg 1 \quad \mathrm{Card}\{j \in \mathbb{N}, \quad \lambda_j \leq \lambda\} = c(d)\mathrm{Vol}(X)\lambda^d + \mathcal{O}(\lambda^{d-1}). \tag{7}$$

An easy consequence is the following lemma.

Lemma 3. — *There are two constants $\alpha \geq 1$ and $C \geq 1$ which only depend on X such that the following holds*

$$\forall k \in \mathbb{N} \qquad \frac{1}{C}(1+k)^{d-1} \leq Card\{j \in \mathbb{N}, \quad \lambda_j \in (\alpha k, \alpha k + \alpha]\} \leq C(1+k)^{d-1}. \tag{8}$$

In particular, the interval $(\alpha k, \alpha k + \alpha]$ contains at least one eigenvalue of $\sqrt{-\Delta}$.

Proof. For the case $k = 0$, (8) merely means that the interval $(0, \alpha]$ contains at least one eigenvalue. This is obvioully true if $\alpha \geq 1$ is large enough. We may now assume that k is greater or equal to 1. One may reformulate (7) in the following way : there are a constant $B > 0$ and a bounded function $\mathcal{B} : [1, +\infty[\to [-B, B]$ such that

$$\forall \lambda \geq 1 \quad \mathrm{Card}\{j \in \mathbb{N}, \quad \lambda_j \leq \lambda\} = c(d)\mathrm{Vol}(X)\left[\lambda^d + \lambda^{d-1}\mathcal{B}(\lambda)\right]$$

Forgetting $c(d)\mathrm{Vol}(X)$, we have to prove

$$\frac{1}{C}k^{d-1} \leq \underbrace{(\alpha k+\alpha)^d - (\alpha k)^d}_{=\Theta_1} + \underbrace{(\alpha k+\alpha)^{d-1}\mathcal{B}(\alpha k+\alpha) - \alpha^{d-1}k^{d-1}\mathcal{B}(\alpha k)}_{=\Theta_2} \leq Ck^{d-1}.$$

We can find $C(d) \geq 1$ large enough such that the following inequalities hold true for any $k \in \mathbb{N}^\star$:

$$\frac{\alpha^d k^{d-1}}{C(d)} \leq \Theta_1 \leq C(d)\alpha^d k^{d-1} \quad \text{and} \quad |\Theta_2| \leq B\alpha^{d-1}[(k+1)^{d-1} + k^{d-1}] \leq BC(d)\alpha^{d-1}k^{d-1}.$$

We now choose $\alpha > 2BC(d)^2$, so we have $|\Theta_2| \leq \frac{\alpha^d k^{d-1}}{2C(d)}$. Combining all the previous inequalities, we get

$$\frac{\alpha^d k^{d-1}}{2C(d)} \leq \Theta_1 + \Theta_2 \leq \left(C(d) + \frac{1}{2C(d)}\right)\alpha^d k^{d-1}.$$

□

It will be convenient to consider from now the increasing sequence $(\mu_j)_{j\geq 0}$ of all the eigenvalues of $\sqrt{-\Delta}$ *without multiplicities.* The Weyl law therefore implies the following asymptotic

$$\forall \lambda \gg 1 \quad \mathrm{Card}\{j \in \mathbb{N}, \quad \mu_j \leq \lambda\} \leq c(X)\lambda^d. \tag{9}$$

Note that Lemma 3 also gives us

$$\forall k \in \mathbb{N} \qquad 1 \leq \mathrm{Card}\{j \in \mathbb{N}, \quad \mu_j \in (\alpha k, \alpha k + \alpha]\} \leq C(1+k)^{d-1}.$$

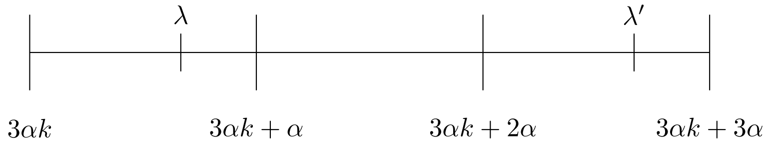
We claim that the previous allows to prove a weak property of separation of the eigenvalues $(\mu_j)_{j\in\mathbb{N}}$.

Proposition 4. — *We denote by α the constant of Lemma 3. There is a constant $C = C(X) \geq 1$ and an increasing sequence of integers $(j_k)_{k\in\mathbb{N}}$ such that the following holds for any $k \in \mathbb{N}$:*

$$3\alpha k \leq \mu_{j_k} \leq 3\alpha(k+1)$$

$$\mu_{1+j_k} - \mu_{j_k} \geq \frac{1}{C\mu_{j_k}^{d-1}} \tag{10}$$

Proof. The intervals $(3\alpha k, 3\alpha k + \alpha]$ and $(3\alpha k + 2\alpha, 3\alpha k + 3\alpha]$ contain each at least one eigenvalue of $\sqrt{-\Delta}$, say respectively λ and λ'.



Moreover, there are at most $C(X)(1+k)^{d-1}$ eigenvalues in $(3\alpha k, 3\alpha k+3\alpha]$. Using the inequality $\lambda+\alpha<\lambda'$, we see that $(3\alpha k, 3\alpha k+3\alpha]$ contains at least two successive eigenvalues μ_{j_k} and μ_{1+j_k} (between λ and λ') such that

$$\mu_{1+j_k}-\mu_{j_k}\geq \frac{\alpha}{C(X)(1+k)^{d-1}}.$$

As μ_{j_k} belongs to $(3\alpha k, 3\alpha k+3\alpha]$, one may replace the lower bound $\frac{\alpha}{C(X)(1+k)^{d-1}}$ by $\frac{1}{C\mu_{j_k}^{d-1}}$ (with a constant $C\geq 1$ which only depends on X). □

Remark 5. — *There is a simple way that allows to obtain the exponent $d-1$ in the lower bound (10). Remember that the equivalent $\mu_k\sim C(X)k^{\frac{1}{d}}$ holds true for a generic metric on X. It is therefore natural to compare $\mu_{k+1}-\mu_k$ to $k^{\frac{1}{d}-1}\simeq \mu_k^{-(d-1)}$. Proposition 4 says that such an heuristic argument does hold true in a statistical sense for any metric on X.*

Let us define the following intervals $I_0:=[0,\mu_{j_0}]$ and

$$I_k:=[\mu_{1+j_{k-1}},\mu_{j_k}]\subset[3\alpha(k-1),3\alpha(k+1)],\qquad \forall k\geq 1. \tag{11}$$

Proposition 4 says that the family of intervals $(I_k)_{k\geq 0}$ has a linear growth in k and that the distance of two successive intervals behaves like a negative power of their range of frequency.

We also define the spectral projector $\Pi_k : L^2(X)\to L^2(X)$ by $\Pi_k=\mathbf{1}_{I_k}(\sqrt{-\Delta})$. As each eigenvalue of $\sqrt{-\Delta}$ belongs to $\bigcup\limits_{k\in\mathbb{N}} I_k$, one may define the Sobolev norm $\|\cdot\|_{H^s(X)}$ of a function $\phi\in H^s(X)$ with the following expression

$$\|\phi\|_{H^s(X)}:=\sqrt{\sum_{k\in\mathbb{N}}(1+k)^{2s}\,\|\Pi_k(\phi)\|^2_{L^2(X)}},\quad s\in\mathbb{R}. \tag{12}$$

Remark 6. — *Note that the construction held in the proof of Proposition 4 does not give the same intervals than those of* [**DS06, BDGS07**]. *Assume for instance that X is the usual torus $\mathbb{T}$, then one has $\mu_j=j$ for each $j\in\mathbb{N}$. In this example, one may choose $\alpha=1$ in Lemma 3 and $\mu_{j_k}=3k$ in Proposition 4. In other words, this construction gives us $I_k=[3k-2,3k]$ for each integer $k\in\mathbb{N}^\star$ whereas one should naturally choose $I_k=\{k\}$. In both cases, the only relevant thing to see is that the length of I_k is uniformly bounded with respect to k.*

In the same spirit that in the paper [**DS06**, Part 2.2] which deals with Zoll manifolds, Proposition 4 allows us to validate the proof of [**DS04**, Theorem 4.7]. We get the following statement.

***Proposition 7* (Delort-Szeftel)**. — *For almost every $m>0$ (in the sense of Lebesgue), there are $N_0>0$ and $C(m,n,N_0,d)>0$ such that the following holds. For any integer $p\in[0,n+2]$, any $(k_1,\dots,k_{n+2})\in\mathbb{N}^{n+2}$ and any $(\xi_{k_1},\dots,\xi_{k_{n+2}})\in(I_{k_1}\times\cdots\times I_{k_{n+2}})\cap(sp(\sqrt{-\Delta})^{n+2}$ (where $sp(\sqrt{-\Delta})$ is the spectrum of $\sqrt{-\Delta}$), exactly one of the following two assertions holds*

i) (resonant regime) n is even, p equals $\frac{n+2}{2}$ and there is a permutation τ of the set $\{1,\dots,p\}$ such that, for every integer $j\in[1,p]$, the equality $k_{\tau(j)}=k_{p+j}$ holds true.

ii) (nonresonant regime) we have

$$\left|\sum_{j=1}^{p}\sqrt{m^2+\xi_{k_j}^2}-\sum_{j=p+1}^{n+2}\sqrt{m^2+\xi_{k_j}^2}\right|\geq\frac{C(m,n,N_0,d)}{(1+\max(\xi_{k_1},\dots,\xi_{k_{n+2}}))^{N_0}}. \tag{13}$$

PROOF. The proof of [**DS04**, Theorem 4.7, see the lines (5.31),(5.32)] only relies on the existence of two positive constants C and c such that the following two statements hold

i) Thanks to (9), we may claim that the cardinal of $\{\xi \in (\mathrm{sp}(\sqrt{-\Delta}))^{n+2}, \quad |\xi| < \lambda\}$ is less than $C\lambda^{(n+2)d}$ for any $\lambda \geq 1$.
ii) Consider two tuples $(k_1,\dots,k_{n+2}) \in \mathbb{N}^{n+2}$ and $(\xi_{k_1},\dots,\xi_{k_{n+2}}) \in (I_{k_1}\times\cdots\times I_{k_{n+2}}) \cap (\mathrm{sp}(\sqrt{-\Delta}))^{n+2}$. If we assume that n is even, p equals $\frac{n+2}{2}$ and that for any permutation τ of the set $\{1,\dots,p\}$, there is at least one integer $j \in [1,p]$ such that $k_{\tau(j)} \neq k_{p+j}$, then Proposition 4 ensures that the following inequality holds true

$$\sum_{j=1}^{p}(\xi^2_{k_{\tau(j)}} - \xi^2_{k_{p+j}})^2 \geq \frac{c}{(1+\max(\xi_{k_1},\dots,\xi_{k_{n+2}})^{2(d-2)}}.$$

□

Remark 8. — *We learn several things by looking at the proof the of* [**DS04**]. *Firstly, the most favorable case occurs if n is odd. In this case, we we do not need any separation of eigenvalues and no need to select eigenvalues in packets. By contrast, if n is even then* (13) *implies that the distance of two eigenvalues which belong to two different packets I_k and $I_{k'}$ is at least comparable to a negative power of their maximum. A property of separation as the one given by Proposition 4 is therefore unavoidable. More precisely, if n is even, the mere case which needs Proposition 4 is $p = \frac{n+2}{2}$ because of simultaneous compensations (see* [**DS04**, Part 5, line (5.39)]*).*

The selection resonant/nonresonant should be compared to (5). In the particular case where each eigenvalue of $-\Delta$ belongs to $\mathbb{N}$, we can state an algebraic counterpart.

Proposition 9. — *Consider a positive real algebraic number m of degree $\deg(m) > 2^{n+1}$. For any real number $N_0 > 2^{2n+1}-1$, there is a constant $C(m,n,N_0) > 0$ such that the following holds. For any integer $p \in [0,n+2]$ and for any $(k_1,\dots,k_{n+2}) \in \mathbb{N}^{n+2}$, exactly one of the following two assertions holds*

i) (resonant regime) n is even, p equals $\frac{n+2}{2}$ and there is a permutation τ of the set $\{1,\dots,p\}$ such that, for every integer $j \in [1,p]$, the equality $k_{\tau(j)} = k_{p+j}$ holds true.
ii) (nonresonant regime) we have

$$\left|\sum_{j=1}^{p}\sqrt{m^2+k_j} - \sum_{j=p+1}^{n+2}\sqrt{m^2+k_j}\right| \geq \frac{C(m,n,N_0)}{(1+\max(k_1,\dots,k_{n+2}))^{\frac{N_0}{2}}}. \tag{14}$$

Proposition 9 will be proved at the end of Section 8.

In sections 3, 4, 5 and 7, we will assume the mass m to be generic in the sense of Proposition 7 or Proposition 9. Moreover, we will denote by N_0 the constant which appears in Proposition 7 or Proposition 9. Notice that according to (11), the right hand side of (13) may be written as the one in (14), with the value of N_0 given in Proposition 7.

3. General strategy of normal form

Let us denote by Λ_m the pseudo-differential operator $\sqrt{-\Delta+m^2}$ and D_t equals $-i\partial_t$ as usual. We reduce the equation (1) by setting $u = -(D_t+\Lambda_m)w$ and $w = -\mathrm{Re}(\Lambda_m^{-1}u) = -\frac{1}{2}\Lambda_m^{-1}u - \frac{1}{2}\Lambda_m^{-1}\overline{u}$. We thus get

$$(D_t - \Lambda_m)u = w^{n+1} = \sum_{p=0}^{n+1} c_{n,p}M_{m,n}(\overbrace{u,\dots,u}^{p},\overbrace{\overline{u},\dots,\overline{u}}^{n+1-p}), \tag{15}$$

where one sets

$$c_{n,p} := \begin{pmatrix} n+1 \\ p \end{pmatrix}\frac{(-1)^{n+1}}{2^{n+1}}, \qquad M_{m,n}(u_1,\dots,u_{n+1}) := (\Lambda_m^{-1}u_1)\dots(\Lambda_m^{-1}u_{n+1}). \tag{16}$$

After such a reduction to the order one, we have to get a priori H^s bounds of u instead of w since we have $\|u\|_{H^s(X)} \simeq \|\partial_t w\|_{H^s(X)} + \|w\|_{H^{s+1}(X)}$. Instead of (12), it is much more convenient to consider the following expression

$$\Theta_s(u) := \frac{1}{2}\sum_{k\in\mathbb{N}}(1+k)^{2s+1}\left\|\Pi_k\Lambda_m^{-\frac{1}{2}}u\right\|^2_{L^2(X)}.$$

We clearly have $\Theta_s(u) \simeq \|u\|^2_{H^s(X)}$ (up to a multiplicative constant which depends on m and X). In the case where I_k does contain only one eigenvalue λ, several computations are easier due to the formula $\Lambda_m\Pi_k = \sqrt{\lambda^2+m^2}\Pi_k$. In the general case that concerns us, the latter equality does not hold true and we can merely use the formula $\Lambda_m\Pi_k = \Pi_k\Lambda_m$. This is why we introduce $\Theta_s(u)$. More precisely, the formula of $\Theta_s(u)$ will indeed provide a very simple computation in the resonant regime (see below (22)). Let us now compute the derivative of $\Theta_s(u)$:

$$\begin{aligned}\frac{d}{dt}\Theta_s(u) &= \sum_{k\in\mathbb{N}}(1+k)^{2s+1}\mathrm{Re}\ \langle\dot{u},\Pi_k\Lambda_m^{-1}u\rangle\\ &= \sum_{k\in\mathbb{N}}(1+k)^{2s+1}\mathrm{Re}\ i\left[\langle\Lambda_m u,\Pi_k\Lambda_m^{-1}u\rangle + \sum_{p=0}^{n+1}c_{n,p}\langle M_{m,n}(\overbrace{u,\dots,u}^{p},\overbrace{\overline{u},\dots,\overline{u}}^{n+1-p}),\Pi_k\Lambda_m^{-1}u\rangle\right]\end{aligned}$$

The equality $\langle\Lambda_m u,\Pi_k\Lambda_m^{-1}u\rangle = \|\Pi_k u\|^2_{L^2(X)}$ allows to vanish the first term and we thus get

$$\frac{d}{dt}\Theta_s(u) = \sum_{k\in\mathbb{N}}\sum_{p=0}^{n+1}(1+k)^{2s+1}c_{n,p}\mathrm{Re}\ i\langle M_{m,n}(\overbrace{u,\dots,u}^{p},\overbrace{\overline{u},\dots,\overline{u}}^{n+1-p}),\Pi_k\Lambda_m^{-1}u\rangle. \tag{17}$$

The strategy of normal form consists in making a perturbation of $\Theta_s(u)$ by a $(n+2)$-homogeneous expression of the form

$$\widetilde{\Theta}_s(u) := \sum_{k\in\mathbb{N}}\sum_{p=0}^{n+1}(1+k)^{2s+1}c_{n,p}\mathrm{Re}\ \langle\widetilde{M}_{m,n,p}(\overbrace{u,\dots,u}^{p},\overbrace{\overline{u},\dots,\overline{u}}^{n+1-p}),\Pi_k\Lambda_m^{-1}u\rangle \tag{18}$$

where $\widetilde{M}_{m,n,p}$ is expected to be a bounded $(n+1)$-multilinear operator from $H^s(X)^{n+1}$ to $H^s(X)$. Using (15) and assuming that the inner product $\langle\cdot,\cdot\rangle$ is antilinear in its second argument, one computes the derivative of (18)

$$\frac{d}{dt}\widetilde{\Theta}_s(u) = \mathcal{N}_{2n+2}(u) + \sum_{k\in\mathbb{N}}\sum_{p=0}^{n+1}(1+k)^{2s+1}c_{n,p}\mathrm{Re}\ i\langle\mathcal{L}_p(\widetilde{M}_{m,n,p})(\overbrace{u,\dots,u}^{p},\overbrace{\overline{u},\dots,\overline{u}}^{n+1-p}),\Pi_k\Lambda_m^{-1}u\rangle, \tag{19}$$

where

- the term $\mathcal{N}_{2n+2}(u)$ is the sum of several $(2n+2)$-homogeneous terms obtained by replacing one of the $n+2$ terms u in (18) by w^{n+1},
- the operator $\mathcal{L}_p$ is defined by the following formula for any $(n+1)$-multilinear operator M and any tuple of functions $(u_1,\dots,u_{n+1})$:

$$\begin{aligned}\mathcal{L}_p(M)(u_1,\dots,u_{n+1}) &= \sum_{j=1}^{p}M(\overbrace{u_1,\dots,u_{j-1},\Lambda_m u_j}^{j},\overbrace{u_{j+1},\dots,u_{n+1}}^{n+1-j})\\ &\quad-\sum_{j=p+1}^{n+1}M(\overbrace{u_1,\dots,u_{j-1},\Lambda_m u_j}^{j},\overbrace{u_{j+1},\dots,u_{n+1}}^{n+1-j})\\ &\quad-\Lambda_m M(u_1,\dots,u_{n+1}).\end{aligned} \tag{20}$$

Such a strategy would be of interest if the derivative of $\Theta_s(u)-\widetilde{\Theta}_s(u)$ could kill each $M_{m,n}$. By comparing (17) and (19), it is sufficient to construct the operator $\widetilde{M}_{m,n,p} : H^s(X)^{n+1}\to H^s(X)$ such that the following holds true for any integers $k \in \mathbb{N}$ and $p \in [0, n+1]$

$$\mathrm{Re}\ i\langle M_{m,n}(\overbrace{u,\dots,u}^{p},\overbrace{\overline{u},\dots,\overline{u}}^{n+1-p}),\Pi_k\Lambda_m^{-1}u\rangle = \mathrm{Re}\ i\langle \mathcal{L}_p(\widetilde{M}_{m,n,p})(\overbrace{u,\dots,u}^{p},\overbrace{\overline{u},\dots,\overline{u}}^{n+1-p}),\Pi_k\Lambda_m^{-1}u\rangle$$

The boundedness of $\widetilde{M}_{m,n,p} : H^s(X)^{n+1} \to H^s(X)$ would imply the estimates

$$\left|\frac{d}{dt}\left(\Theta_s(u) - \widetilde{\Theta}_s(u)\right)\right| = |\mathcal{N}_{2n+2}(u)| \lesssim \|u\|_{H^s(X)}^{2n+2},$$

and we would consequently be able to improve the local existence time to ε^{-2n} thanks to an a priori estimate on $\Theta_s(u) - \widetilde{\Theta}_s(u)$ (which is of the same order than $\Theta_s(u)$ for small solutions).

Let us explain how to construct, at least formally, $\widetilde{M}_{m,n,p}$ and why we are unable to prove its H^s-boundedness. The most natural way to define the operator $\widetilde{M}_{m,n,p}$ is to spectrally decompose u and $\overline{u}$ and to solve, for any $(k_1, \dots, k_{n+2}) \in \mathbb{N}^{n+2}$, the following *homological equation*

$$\begin{aligned} &\mathrm{Re}\ i\langle M_{m,n}(\Pi_{k_1}u,\dots,\Pi_{k_p}u,\Pi_{k_{p+1}}\overline{u},\dots,\Pi_{k_{n+1}}\overline{u}),\Pi_{k_{n+2}}\Lambda_m^{-1}u\rangle \\ &= \mathrm{Re}\ i\langle \mathcal{L}_p(\widetilde{M}_{m,n,p})(\Pi_{k_1}u,\dots,\Pi_{k_p}u,\Pi_{k_{p+1}}\overline{u},\dots,\Pi_{k_{n+1}}\overline{u}),\Pi_{k_{n+2}}\Lambda_m^{-1}u\rangle \end{aligned} \tag{21}$$

We claim that the previous equality is easy to satisfy in the resonant regime. Let us denote by $\mathcal{R}_{n+2,p} \subset \mathbb{N}^{n+2}$ the subset of tuples $(k_1, \dots, k_{n+2})$ which are resonant (see condition i of Proposition 7 and 9). For any $(k_1, \dots, k_{n+2}) \in \mathcal{R}_{n+2,p}$, it is clear that the definition (16) implies the following

$$\mathrm{Re}\ i\langle M_{m,n}(\Pi_{k_1}u,\dots,\Pi_{k_p}u,\Pi_{k_{p+1}}\overline{u},\dots,\Pi_{k_{n+1}}\overline{u}),\Pi_{k_{n+2}}\Lambda_m^{-1}u\rangle = \mathrm{Re}\ i\int_X \prod_{j=1}^{p} |\Pi_{k_j}\Lambda_m^{-1}u|^2 dx = 0 \tag{22}$$

Let us also denote by E_k the range of the spectral projector Π_k for any $k \in \mathbb{N}$. As Λ_m and Π_k commute, one has the inclusion $\Lambda_m(E_k) \subset E_k$. Because of (22),(21) and (20), it is sufficient to define

$$\forall (k_1,\dots,k_{n+2}) \in \mathcal{R}_{n+2,p} \qquad \forall (u_1,\dots,u_{n+1}) \in E_{k_1}\times\dots\times E_{k_{n+1}} \quad \Pi_{k_{n+2}}\widetilde{M}_{m,n,p}(u_1,\dots,u_{n+1}) = 0.$$

We now have to solve (21) for nonresonant tuples $(k_1, \dots, k_{n+2}) \in \mathbb{N}^{n+2}\backslash\mathcal{R}_{n+2,p}$. For clarity, we begin by the case where each interval I_k contains only one eigenvalue, say μ_k. The formula $\Pi_k\Lambda_m = \sqrt{\mu_k^2 + m^2}\Pi_k$ allows to reduce the equation (21) to the following

$$\begin{aligned} &\mathrm{Re}\ i\langle M_{m,n}(\Pi_{k_1}u,\dots,\Pi_{k_p}u,\Pi_{k_{p+1}}\overline{u},\dots,\Pi_{k_{n+1}}\overline{u}),\Pi_{k_{n+2}}\Lambda_m^{-1}u\rangle \\ &= F_{m,n,p}(\mu_{k_1},\dots,\mu_{k_{n+2}})\mathrm{Re}\ i\langle \widetilde{M}_{m,n,p}(\Pi_{k_1}u,\dots,\Pi_{k_p}u,\Pi_{k_{p+1}}\overline{u},\dots,\Pi_{k_{n+1}}\overline{u}),\Pi_{k_{n+2}}\Lambda_m^{-1}u\rangle \end{aligned}$$

with

$$F_{m,n,p}(\mu_{k_1},\dots,\mu_{k_{n+2}}) := \sum_{j=1}^{p}\sqrt{\mu_{k_j}^2 + m^2} - \sum_{j=p+1}^{n+2}\sqrt{\mu_{k_j}^2 + m^2}.$$

A natural choice for the operator $\widetilde{M}_{m,n,p}$ is therefore

$$\widetilde{M}_{m,n,p}(u_1,\dots,u_{n+1}) = \sum_{k\in\mathbb{N}^{n+2}\backslash\mathcal{R}_{n+2,p}} \frac{\Pi_{k_{n+2}}M_{m,n}(\Pi_{k_1}u_1,\dots,\Pi_{k_{n+1}}u_{n+1})}{F_{m,n,p}(\mu_{k_1},\dots,\mu_{k_{n+2}})}, \tag{23}$$

for any $(u_1, \dots, u_{n+1}) \in H^s(X)^{n+1}$. With a very weak property of separation (see Proposition 4), the best known estimates are given by Proposition 7 and Proposition 9 :

$$\frac{1}{|F_{m,n,p}(\mu_{k_1},\dots,\mu_{k_{n+2}})|} \le C(m,n,N_0,d)(1+\max(\mu_{k_1},\dots,\mu_{k_{n+2}}))^{N_0}. \tag{24}$$

With such inequalities, we do not know if the operator $\widetilde{M}_{m,n,p}$ is bounded on $H^s(X)^{n+1}$ to $H^s(X)$. In Section 5, we explain how to modify this strategy and we will do a *partial normal form.* In other words, we will merely eliminate a part of each term $M_{m,n}$.

We end this part by explaining how to deal with the general case where I_k may contain several eigenvalues. For the sake of clarity, we introduce a notation all along this part. Considering $n+2$ integers $k_1, \dots, k_{n+2}$, we denote by $k_1^\star \ge k_2^\star \ge k_3^\star$ the three largest ones among $1+k_1, \dots, 1+k_{n+2}$. For instance

$$k_1^\star = 1 + \max(k_1, \dots, k_{n+2}). \tag{25}$$

The following result has to be seen has an analogue of [**DS06**, Proposition 2.4] but our proof is much simpler because we may authorize a loss of a power of the largest frequency (this was forbidden in [**DS06**]).

Proposition 10. — *Consider an integer $p \in [0, n+1]$ and a tuple $(k_1, \dots, k_{n+2}) \in \mathbb{N}^{n+2} \backslash \mathcal{R}_{n+2,p}$. Denote by Mult$(E_{k_1} \times \cdots \times E_{k_{n+1}}, E_{k_{n+2}})$ the finite-dimensional vector space of multilinear operators $M : E_{k_1} \times \cdots \times E_{k_{n+1}} \to E_{k_{n+2}}$. We endow Mult$(E_{k_1} \times \cdots \times E_{k_{n+1}}, E_{k_{n+2}})$ with its natural norm. The linear operator defined in* (20)

$$\mathcal{L}_p : \mathit{Mult}(E_{k_1} \times \cdots \times E_{k_{n+1}}, E_{k_{n+2}}) \to \mathit{Mult}(E_{k_1} \times \cdots \times E_{k_{n+1}}, E_{k_{n+2}})$$

is invertible and its inverse fulfills the estimate

$$\left\|\mathcal{L}_p^{-1}\right\| \leq C(m, n, N_0, X)(k_1^\star)^{N_0 + \frac{d-1}{2}} (k_3^\star)^{\frac{n(d-1)}{2}}.$$

The previous result allows to define a global inverse of $\mathcal{L}_p$. The most rigorous way to define such an operator is to introduce the algebraic sum

$$E_\infty := \bigoplus_{k \in \mathbb{N}} E_k,$$

which can be seen has a dense vectorial subspace of $L^2(X)$. Let us denote by $\mathcal{D}'(X)$ the vectorial space of distributions on X. For any multilinear operator $M : E_\infty^{n+1} \to \mathcal{D}'(X)$ and for any $(k_1, \dots, k_{n+2}) \in \mathbb{N}^{n+2} \backslash \mathcal{R}_{n+2,p}$, the operator $\Pi_{k_{n+2}} M(\Pi_{k_1} \bullet, \dots, \Pi_{k_{n+1}} \bullet)$ is well defined as an element of Mult$(E_{k_1} \times \cdots \times E_{k_{n+1}}, E_{k_{n+2}})$. We may thus define for any $(u_1, \dots, u_{n+1}) \in E_{k_1} \times \cdots \times E_{k_{n+1}}$:

$$\left[\Pi_{k_{n+2}} \mathcal{L}_p^{-1}(M)\right](u_1, \dots, u_{n+1}) := \mathcal{L}_p^{-1}\left(\Pi_{k_{n+2}} M(\Pi_{k_1} \bullet, \dots, \Pi_{k_{n+1}} \bullet)\right)(u_1, \dots, u_{n+1}), \tag{26}$$

The previous formula allows us to define

$$\forall (u_1, \dots, u_{n+1}) \in E_\infty^{n+1} \quad \mathcal{L}_p^{-1}(M)(u_1, \dots, u_{n+1}) := \sum_{k \in \mathbb{N}^{n+2} \backslash \mathcal{R}_{n+2,p}} \left[\Pi_{k_{n+2}} \mathcal{L}_p^{-1}(M)\right](\Pi_{k_1} u_1, \dots, \Pi_{k_{n+1}} u_{n+1}). \tag{27}$$

In other words, if one wants to solve (21) then Proposition 10 says that the good candidate for $\widetilde{M}_{m,n,p}$ is $\mathcal{L}_p^{-1}(M_{m,n})$ (instead of (23)). To finish this part, we note that the formula (27) is essentially formal and we need to study its convergence.

4. Proof of Proposition 10

For any integer $j \in [1, n+2]$, remember that E_{k_j} is the range of $\Pi_{k_j} = \mathbf{1}_{I_{k_j}}(\sqrt{-\Delta})$. Let us denote by $(\phi_{k_j,\ell_j})_{\ell_j}$ an orthonormal basis of eigenfunctions of $\sqrt{-\Delta}$ on E_{k_j} (the integer ℓ_j runs over $[1, \dim(E_{k_j})] \cap \mathbb{N}$). We also write $\sqrt{-\Delta}\phi_{k_j,\ell_j} = \lambda_{k_j,\ell_j}\phi_{k_j,\ell_j}$ with $\lambda_{k_j,\ell_j} \in I_{k_j}$. Let us begin the proof :

$$\forall M \in \mathrm{Mult}(E_{k_1} \times \cdots \times E_{k_{n+1}}, E_{k_{n+2}}), \quad \forall (u_1, \dots, u_{n+1}) \in E_{k_1} \times \cdots \times E_{k_{n+1}}$$

$$M(u_1, \dots, u_{n+1}) = \sum_{\substack{1 \leq \ell_1 \leq \dim(E_{k_1}) \\ \cdots \\ 1 \leq \ell_{n+2} \leq \dim(E_{k_{n+2}})}} \langle M(\phi_{k_1,\ell_1}, \dots, \phi_{k_{n+1},\ell_{n+1}}), \phi_{k_{n+2},\ell_{n+2}} \rangle \phi_{k_{n+2},\ell_{n+2}} \prod_{j=1}^{n+1} \langle u_j, \phi_{k_j,\ell_j} \rangle.$$

The definition (20) of $\mathcal{L}_p$ ensures that $\mathcal{L}_p(M)(u_1, \dots, u_{n+1})$ equals

$$\sum_{\substack{1 \leq \ell_1 \leq \dim(E_{k_1}) \\ \cdots \\ 1 \leq \ell_{n+2} \leq \dim(E_{k_{n+2}})}} F_{m,n,p}(\lambda_{k_1,\ell_1}, \dots, \lambda_{k_{n+2},\ell_{n+2}}) \langle M(\phi_{k_1,\ell_1}, \dots, \phi_{k_{n+1},\ell_{n+1}}), \phi_{k_{n+2},\ell_{n+2}} \rangle \phi_{k_{n+2},\ell_{n+2}} \prod_{j=1}^{n+1} \langle u_j, \phi_{k_j,\ell_j} \rangle.$$

The previous formula may be rephrased as follows : the operator $\mathcal{L}_p$ is diagonalizable on Mult$(E_{k_1} \times \cdots \times E_{k_{n+1}}, E_{k_{n+2}})$, and for any $(\ell_1, \dots, \ell_{n+2})$, the multilinear operator

$$(u_1, \dots, u_{n+1}) \mapsto \phi_{k_{n+2},\ell_{n+2}} \prod_{j=1}^{n+1} \langle u_j, \phi_{k_j,\ell_j} \rangle,$$

is nothing else than an eigenvector of $\mathcal{L}_p$ associated to the eigenvalue $F_{m,n,p}(\lambda_{k_1,\ell_1},\dots,\lambda_{k_{n+2},\ell_{n+2}})$. Remember that we are in a nonresonant regime, so each eigenvalue $F_{m,n,p}(\lambda_{k_1,\ell_1},\dots,\lambda_{k_{n+2},\ell_{n+2}})$ is nonzero. It is thus clear that $\mathcal{L}_p$ is invertible and that $\mathcal{L}_p^{-1}(M)(u_1,\dots,u_{n+1})$ equals

$$\sum_{\substack{1\le \ell_1\le \dim(E_{k_1})\\ \cdots \\ 1\le \ell_{n+2}\le \dim(E_{k_{n+2}})}} \frac{\langle M(\phi_{k_1,\ell_1},\dots,\phi_{k_{n+1},\ell_{n+1}}),\phi_{k_{n+2},\ell_{n+2}}\rangle}{F_{m,n,p}(\lambda_{k_1,\ell_1},\dots,\lambda_{k_{n+2},\ell_{n+2}})}\phi_{k_{n+2},\ell_{n+2}}\prod_{j=1}^{n+1}\langle u_j,\phi_{k_j,\ell_j}\rangle.$$

In order to make our arguments symmetric, it is convenient to remark that for any $u_{n+2}\in E_{k_{n+2}}$, the scalar product $\langle \mathcal{L}_p^{-1}(M)(u_1,\dots,u_{n+1}),u_{n+2}\rangle$ equals

$$\sum_{\substack{1\le \ell_1\le \dim(E_{k_1})\\ \cdots \\ 1\le \ell_{n+2}\le \dim(E_{k_{n+2}})}} \frac{\langle M(\phi_{k_1,\ell_1},\dots,\phi_{k_{n+1},\ell_{n+1}}),\phi_{k_{n+2},\ell_{n+2}}\rangle}{F_{m,n,p}(\lambda_{k_1,\ell_1},\dots,\lambda_{k_{n+2},\ell_{n+2}})}\prod_{j=1}^{n+2}\langle u_j,\phi_{k_j,\ell_j}\rangle.$$

Combining (24) and the Cauchy-Schwarz inequality, we get the following upper bound, up to a multiplicative constant $C(m,n,N_0,X)$, of $|\langle \mathcal{L}_p^{-1}(M)(u_1,\dots,u_{n+1}),u_{n+2}\rangle|$

(28)

$$(1+\max(k_1,\dots,k_{n+2}))^{N_0}\left(\sum_{\substack{1\le \ell_1\le \dim(E_{k_1})\\ \cdots \\ 1\le \ell_{n+2}\le \dim(E_{k_{n+2}})}} |\langle M(\phi_{k_1,\ell_1},\dots,\phi_{k_{n+1},\ell_{n+1}}),\phi_{k_{n+2},\ell_{n+2}}\rangle|^2\right)^{\frac12}\prod_{j=1}^{n+2}\|u_j\|_{L^2(X)}$$

Without loss of generality, we may assume that $k_{n+2}\ge\cdots\ge k_1$ holds true (the following arguments are still available without this assumption). Using the linearity with respect to the k_{n+2}th variable and the fact that $(\phi_{k_{n+2},\ell_{n+2}})_{\ell_{n+2}}$ is an orthonormal basis of $E_{k_{n+2}}$, we can write for any fixed $(\ell_1,\dots,\ell_{n+1})$:

$$\sum_{\ell_{n+2}=1}^{\dim(E_{k_{n+2}})}|\langle M(\phi_{k_1,\ell_1},\dots,\phi_{k_{n+1},\ell_{n+1}}),\phi_{k_{n+2},\ell_{n+2}}\rangle|^2=\sup_{\substack{u_{n+2}\in E_{k_{n+2}}\\ \|u_{n+2}\|_{L^2(X)}=1}}\left|\langle M(\phi_{k_1,\ell_1},\dots,\phi_{k_{n+1},\ell_{n+1}}),u_{n+2})\rangle\right|^2.$$

Since the previous bound is clearly less or equal to $\|M\|^2$, we can bound (28) by

$$(1+k_{n+2})^{N_0}\sqrt{\dim(E_{k_1})\times\cdots\times\dim(E_{k_{n+1}})}\,\|M\|\prod_{j=1}^{n+2}\|u_j\|_{L^2(X)}.$$

Remember that I_k is a subinterval of $[3\alpha(k-1),3\alpha(k+1)]$ for any $k\in\mathbb{N}^\star$ (see (11)). Forgetting u_{n+2} and using Lemma 3, we get the bound

$$\left\|\mathcal{L}_p^{-1}(M)(u_1,\dots,u_{n+1})\right\|_{L^2(X)}\lesssim(1+k_{n+2})^{N_0+\frac{d-1}{2}}(1+k_n)^{\frac{n(d-1)}{2}}\|M\|\prod_{j=1}^{n+1}\|u_j\|_{L^2(X)}$$

up to a multiplicative constant $C(m,n,N_0,X)$.

5. Strategy of partial normal form

We introduce a cut-off function $\chi:\mathbb{R}\to[0,+\infty[$ fixed once and for all. We may assume that the following holds true for all $\eta\in\mathbb{R}$

$$\begin{aligned}|\eta|\le 1 \quad&\Rightarrow\quad \chi(\eta)=1,\\ 1<|\eta|<2 \quad&\Rightarrow\quad 0<\chi(\eta)<1,\\ 2\le|\eta| \quad&\Rightarrow\quad \chi(\eta)=0.\end{aligned}$$

Let us fix a parameter $\delta>0$ that will be chosen at the end and consider a real number $\varepsilon\in(0,1)$. Writing $u=\chi(\varepsilon^\delta\Lambda_m)u+(1-\chi)(\varepsilon^\delta\Lambda_m)u$, we then decompose $M_{m,n}$ as the sum of two terms :

$$M_{m,n}(\overbrace{u,\dots,u}^{p},\overbrace{\overline{u},\dots,\overline{u}}^{n+1-p})=M_{m,n,\varepsilon,\delta}(\overbrace{u,\dots,u}^{p},\overbrace{\overline{u},\dots,\overline{u}}^{n+1-p})+R_{m,n,p,\varepsilon,\delta}(u),\tag{29}$$

where $M_{m,n,\varepsilon,\delta}$ and $R_{m,n,p,\varepsilon,\delta}(u)$ are defined by

$$
\begin{aligned}
M_{m,n,\varepsilon,\delta}(u_1,\ldots,u_{n+1}) &:= (\Lambda_m^{-1}\chi(\varepsilon^\delta\Lambda_m)u_1))\times\cdots\times(\Lambda_m^{-1}\chi(\varepsilon^\delta\Lambda_m)u_{n+1}))\\
R_{m,n,p,\varepsilon,\delta}(u) &:= \sum_{\Upsilon}\left(\prod_{j=1}^{p}\Lambda_m^{-1}\chi_j(\varepsilon^\delta\Lambda_m)u\right)\left(\prod_{j=p+1}^{n+1}\Lambda_m^{-1}\chi_j(\varepsilon^\delta\Lambda_m)\overline{u}\right),
\end{aligned}
\tag{30}
$$

and $(\chi_1,\ldots,\chi_{n+2})$ runs over the set $\Upsilon := \{\chi, 1-\chi\}^{n+1}\backslash\{(\chi,\ldots,\chi)\}$ of size $2^{n+1}-1$ (in other words, at least one of the functions χ_j equals $1-\chi$). The following lemma will be proved below.

Lemma 11. — *For any $s > \frac{\dim(X)}{2}$, any $u \in H^s(X)$ and any couple $(\epsilon,\delta)\in(0,1)\times(0,+\infty)$, the following holds :*

$$\|R_{m,n,p,\varepsilon,\delta}(u)\|_{H^s(X)} \leq C(m,n,s,X)\varepsilon^\delta \|u\|_{H^s(X)}^{n+1}.$$

The following consequently holds

$$\left|\sum_{k\in\mathbb{N}}(1+k)^{2s+1} Re\ i\langle R_{m,n,p,\varepsilon,\delta}(u), \Pi_k\Lambda_m^{-1}u\rangle\right| \leq C(m,n,s,X)\varepsilon^\delta\|u\|_{H^s(X)}^{n+2}. \tag{31}$$

Combining (17) and (29), the previous lemma allows us to write

$$\frac{d}{dt}\Theta_s(u) = \sum_{k\in\mathbb{N}}\sum_{p=0}^{n+1}(1+k)^{2s+1}c_{n,p}\mathrm{Re}\ i\langle M_{m,n,\varepsilon,\delta}(\overbrace{u,\ldots,u}^{p},\overbrace{\overline{u},\ldots,\overline{u}}^{n+1-p}),\Pi_k\Lambda_m^{-1}u\rangle + \mathcal{O}_{m,n,s,X}(\varepsilon^\delta\|u\|_{H^s(X)}^{n+2}). \tag{32}$$

We now eliminate $M_{m,n,\varepsilon,\delta}$ by a normal form.

Proposition 12. — *There is $s_0 = s_0(d,n,m) > 0$ such that for any $s\in(s_0,+\infty)$, any $\varepsilon\in(0,1)$, any $\delta>0$, the series (27) which defines the operator $\mathcal{L}_p^{-1}(M_{m,n,\varepsilon,\delta})$ converges in the Banach space of bounded $(n+1)$-multilinear operators from $H^s(X)^{n+1}$ to $H^s(X)$. Moreover, we have the estimate*

$$\left\|\mathcal{L}_p^{-1}(M_{m,n,\varepsilon,\delta})(u_1,\ldots,u_{n+1})\right\|_{H^s(X)} \leq C(m,n,X,s)\varepsilon^{-\delta(N_0+\frac{d-1}{2})}\prod_{j=1}^{n+1}\|u_j\|_{H^s(X)}, \tag{33}$$

for any $(u_1,\ldots,u_{n+1})\in H^s(X)^{n+1}$. Adding one more function $u_{n+2}\in H^s(X)$, the following holds

$$\left|\sum_{k\in\mathbb{N}}(1+k)^{2s+1}Re\ \langle\mathcal{L}_p^{-1}(M_{m,n,\varepsilon,\delta})(u_1,\ldots,u_{n+1}),\Lambda_m^{-1}\Pi_k u_{n+2}\rangle\right| \leq C(m,n,X,s)\varepsilon^{-\delta(N_0+\frac{d-1}{2})}\prod_{j=1}^{n+2}\|u_j\|_{H^s(X)}. \tag{34}$$

We can now prove Theorem 1 and 2. The regularity s is assumed to be larger than $\max\left(\frac{\dim(X)}{2}, s_0\right)$. Assume that u is a solution of (15), so one has $\dot{u} = i\Lambda_m u + iw^{n+1}$. Note that the inequalities

$$
\begin{aligned}
\left\|w^{n+1}\right\|_{H^s(X)} &\leq C(m,n,X,s)\|u\|_{H^{s-1}(X)}^{n+1}\\
&\leq C(m,n,X,s)\|u\|_{H^s(X)}^{n+1}
\end{aligned}
\tag{35}
$$

obviously hold true since $H^s(X)$ is an algebra and since w equals $-\mathrm{Re}(\Lambda_m^{-1}u)$. Assume that the initial data $u(0)$ satisfies $\Theta_s(u(0))\leq\varepsilon^2$ and consider the upper bound of the set of the real numbers $t_{\max}\geq 0$ such that $\Theta_s(u(t))\leq 2\varepsilon^2$ holds true for any $t\in[0,t_{\max}]$. As explained in Section 3 (see (19)), one has

$$
\begin{aligned}
&\frac{d}{dt}\sum_{k\in\mathbb{N}}(1+k)^{2s+1}\mathrm{Re}\ \langle \mathcal{L}_p^{-1}(M_{m,n,\varepsilon,\delta})(\overbrace{u,\dots,u}^{p},\overbrace{\overline{u},\dots,\overline{u}}^{n+1-p}),\Lambda_m^{-1}\Pi_k u\rangle \\
&\quad=\sum_{k\in\mathbb{N}}(1+k)^{2s+1}\mathrm{Re}\ i\langle M_{m,n,\varepsilon,\delta}(\overbrace{u,\dots,u}^{p},\overbrace{\overline{u},\dots,\overline{u}}^{n+1-p}),\Lambda_m^{-1}\Pi_k u\rangle \\
&\qquad+\sum_{k\in\mathbb{N}}\sum_{j=1}^{p}(1+k)^{2s+1}\mathrm{Re}\ i\langle \mathcal{L}_p^{-1}(M_{m,n,\varepsilon,\delta})(\overbrace{u,\dots,u,iw^{n+1}}^{j},\overbrace{u,\dots,u}^{p-j},\overbrace{\overline{u},\dots,\overline{u}}^{n+1-p}),\Lambda_m^{-1}\Pi_k u\rangle \\
&\qquad+\sum_{k\in\mathbb{N}}\sum_{j=p+1}^{n+1}(1+k)^{2s+1}\mathrm{Re}\ i\langle \mathcal{L}_p^{-1}(M_{m,n,\varepsilon,\delta})(\overbrace{u,\dots,u}^{p},\overbrace{\overline{u},\dots,-i\overline{w}^{n+1}}^{j-p},\overbrace{\overline{u},\dots,\overline{u}}^{n+1-j},\Lambda_m^{-1}\Pi_k u\rangle \\
&\qquad+\sum_{k\in\mathbb{N}}(1+k)^{2s+1}\mathrm{Re}\ i\langle \mathcal{L}_p^{-1}(M_{m,n,\varepsilon,\delta})(\overbrace{u,\dots,u}^{p},\overbrace{\overline{u},\dots,\overline{u}}^{n+1-p}),i\Lambda_m^{-1}\Pi_k w^{n+1}\rangle.
\end{aligned}
$$

Thanks to (32), (35) and Proposition 12, we can ensure that

$$
\frac{d}{dt}\left[\Theta_s(u)-\sum_{k\in\mathbb{N}}\sum_{p=0}^{n+1}(1+k)^{2s+1}c_{n,p}\mathrm{Re}\ \langle \mathcal{L}_p^{-1}(M_{m,n,\varepsilon,\delta})(\overbrace{u,\dots,u}^{p},\overbrace{\overline{u},\dots,\overline{u}}^{n+1-p}),\Lambda_m^{-1}\Pi_k u\rangle\right] \tag{36}
$$

is $\mathcal{O}_{m,n,X,s}\left(\varepsilon^{\delta+n+2}+\varepsilon^{2n+2-\delta(N_0+\frac{d-1}{2})}\right)$. The previous upper bound is minimal if one chooses $\delta:=\frac{n}{N_0+\frac{d+1}{2}}$. Moreover, for any $t\in[0,t_{\max}]$, Proposition 12 gives us the following estimate (up to a multiplicative constant $C(m,n,X,s)$)

$$
\begin{aligned}
\left|\sum_{k\in\mathbb{N}}\sum_{p=0}^{n+1}(1+k)^{2s+1}c_{n,p}\mathrm{Re}\ \langle \mathcal{L}_p^{-1}(M_{m,n,\varepsilon,\delta})(\overbrace{u(t),\dots,u(t)}^{p},\overbrace{\overline{u(t)},\dots,\overline{u(t)}}^{n+1-p}),\Pi_k\Lambda_m^{-1}u(t)\rangle\right| &\lesssim \varepsilon^{n+2-\delta(N_0+\frac{d-1}{2})}\\
&\lesssim \varepsilon^{2+\frac{n}{N_0+\frac{d+1}{2}}}.
\end{aligned}
$$

The only thing to remark is that the previous bound is negligible with respect to ε^2 if ε is small enough. This fact means that on $[0,t_{\max}]$, the following two quantities are of order ε^2 :

$$
\Theta_s(u(t))\quad\text{and}\quad \Theta_s(u(t))-\sum_{k\in\mathbb{N}}\sum_{p=0}^{n+1}(1+k)^{2s+1}c_{n,p}\mathrm{Re}\ i\langle \mathcal{L}_p^{-1}(M_{m,n,\varepsilon,\delta})(\overbrace{u(t),\dots,u(t)}^{p},\overbrace{\overline{u(t)},\dots,\overline{u(t)}}^{n+1-p}),\Pi_k\Lambda_m^{-1}u(t)\rangle.
$$

By integrating (36) on $[0,t_{\max}]$, we thus get

$$
\begin{aligned}
\varepsilon^2 &\le C(m,n,X,s)t_{\max}\varepsilon^{n+2+\frac{n}{N_0+\frac{d+1}{2}}},\\
\varepsilon^{-n\left(1+\frac{1}{N_0+\frac{d+1}{2}}\right)} &\le C(m,n,X,s)t_{\max}.
\end{aligned} \tag{37}
$$

Similar arguments work for negative times. This finishes the proofs of Theorem 1 and Theorem 2.

6. Proof of Lemma 11

Let us denote by $(\Pi^\bullet_k)_{k\geq 0}$ the sequence of the spectral projectors associated to the sequence of eigenvalues $(\mu_k)_{k\geq 0}$. We clearly have

$$\begin{aligned}
\Lambda_m^{-1}(1-\chi)(\varepsilon^\delta\Lambda_m)u &= \sum_{k\in\mathbb{N}} \frac{(1-\chi)(\varepsilon^\delta\sqrt{\mu_k^2+m^2})}{\sqrt{\mu_k^2+m^2}}\Pi^\bullet_k(u)\\
\left\|\Lambda_m^{-1}(1-\chi)(\varepsilon^\delta\Lambda_m)u\right\|^2_{H^s(X)} &\leq \sum_{k\in\mathbb{N}} \frac{(1+\mu_k)^{2s}}{\mu_k^2+m^2}\left\|\Pi^\bullet_k(u)\right\|^2_{L^2(X)}\mathbf{1}_{(\varepsilon^{-\delta},+\infty)}\left(\sqrt{\mu_k^2+m^2}\right)\\
&\lesssim_{s,X} \varepsilon^{2\delta}\sum_{k\in\mathbb{N}}(1+\mu_k)^{2s}\left\|\Pi^\bullet_k(u)\right\|^2_{L^2(X)}\\
&\lesssim_{s,X} \varepsilon^{2\delta}\left\|u\right\|^2_{H^s(X)}\\
\left\|\Lambda_m^{-1}(1-\chi)(\varepsilon^\delta\Lambda_m)u\right\|_{H^s(X)} &\lesssim_{s,X} \varepsilon^{\delta}\left\|u\right\|_{H^s(X)}.
\end{aligned}$$

Moreover, we have

$$\left\|\Lambda_m^{-1}\chi(\varepsilon^\delta\Lambda_m)u\right\|_{H^s(X)} \leq \left\|\Lambda_m^{-1}u\right\|_{H^s(X)} \leq C(m,s,X)\left\|u\right\|_{H^s(X)}.$$

Using that $H^s(X)$ is an algebra and that ε^δ belongs to $(0,1)$, we obtain the following estimate

$$\left\|R_{m,n,p,\varepsilon,\delta}(u)\right\|_{H^s(X)} \leq C(m,n,s,X)\varepsilon^\delta\left\|u\right\|^{n+1}_{H^s(X)}.$$

The end of the proof is straightforward

$$\begin{aligned}
&\sum_{k\in\mathbb{N}}(1+k)^{2s+1}|\mathrm{Re}\ i\langle R_{m,n,p,\varepsilon,\delta}(u),\Pi_k\Lambda_m^{-1}u\rangle|\\
&\quad\leq \sum_{k\in\mathbb{N}}(1+k)^s\left\|\Pi_kR_{m,n,p,\varepsilon,\delta}(u)\right\|_{L^2(X)}(1+k)^{s+1}\left\|\Pi_k\Lambda_m^{-1}u\right\|_{L^2(X)}\\
&\quad\leq C(m,n,s,X)\varepsilon^\delta\left\|u\right\|^{n+1}_{H^s(X)}\left\|\Lambda_m^{-1}u\right\|_{H^{s+1}(X)}\\
&\quad\leq C(m,n,s,X)\varepsilon^\delta\left\|u\right\|^{n+2}_{H^s(X)}.
\end{aligned}$$

7. Proof of Proposition 12

Assuming (33) the proof of (34) is similar to that of (31), and the proof of (33) will be a consequence of Proposition 16 and Proposition 17.

In the sequel, we will use the notations $k_1^\star, k_2^\star, k_3^\star$ introduced in (25). Using that the range of each spectral projector Π_k is the subspace of functions whose frequencies lie in $I_k\subset[3\alpha k-3\alpha, 3\alpha k+3\alpha]$, we can state the estimates proved in [**DS06**, Proposition 1.2.1].

Proposition 13. — *There are a real number $\nu=\nu(n,d)>0$ and, for any interger $N\geq 0$, a real number $C(n,X,N)>0$ such that for any $(u_1,\dots,u_{n+1})\in L^2(X)^{n+1}$, any nonnegative integers $(k_1,\dots,k_{n+2})\in\mathbb{N}^{n+2}$ the following inequality holds*

$$\left\|\Pi_{k_{n+2}}\left(\Pi_{k_1}(u_1)\dots\Pi_{k_{n+1}}(u_{n+1})\right)\right\|_{L^2(X)} \leq C(n,X,N)\frac{(k_3^\star)^{\nu+N}}{(k_1^\star-k_2^\star+k_3^\star)^N}\prod_{j=1}^{n+1}\left\|u_j\right\|_{L^2(X)}. \tag{38}$$

Remark 14. — *The previous statement is exactly the same than* [**DS06**, Proposition 1.2.1] *thanks to the self-adjointness of the spectral projector $\Pi_{k_{n+2}}$ and the formula*

$$\left\|\Pi_{k_{n+2}}\left(\Pi_{k_1}(u_1)\dots\Pi_{k_{n+1}}(u_{n+1})\right)\right\|_{L^2(X)} = \sup_{\substack{u_{n+2}\in L^2(X)\\ \|u_{n+2}\|_{L^2(X)}=1}}\left|\int_X\Pi_{k_1}(u_1)\dots\Pi_{k_{n+2}}(u_{n+2})dx\right|,$$

where dx is the Riemannian volume on X.

Corollary 15. — *For any $\varepsilon \in (0,1)$, any integer $N \geq 0$, any tuple $(k_1, \dots, k_{n+2}) \in \mathbb{N}^{n+2} \backslash \mathcal{R}_{n+2,p}$ (see (23)), and for any $(u_1, \dots, u_{n+2}) \in L^2(X)^{n+2}$ the following inequality holds true*

$$\begin{aligned}&\left\| \left[\Pi_{k_{n+2}} \mathcal{L}_p^{-1}(M_{m,n,\varepsilon,\delta})\right] (\Pi_{k_1} u_1, \dots, \Pi_{k_{n+1}} u_{n+1}) \right\|_{L^2(X)} \\ &\quad \leq C(m,n,X,N,\delta) \frac{(k_1^\star)^{N_0 + \frac{d-1}{2}} (k_3^\star)^{\nu+N}}{(k_1^\star - k_2^\star + k_3^\star)^N} \left(\prod_{j=1}^{n+1} \frac{\mathbf{1}_{[0,\beta\varepsilon^{-\delta}]}(1+k_j)}{1+k_j} \right) \left(\prod_{j=1}^{n+1} \left\| \Pi_{k_j} u_j \right\|_{L^2(X)} \right),\end{aligned}$$

where N_0 is the constant which appears in (24), $\beta > 0$ is a constant which merely depends on X, and ν is a a constant which merely depends on d and n.

Proof. Thanks to (38) and (30), we can bound

$$\begin{aligned}\left\| \Pi_{k_{n+2}} M_{m,n,\varepsilon,\delta} (\Pi_{k_1} u_1, \dots, \Pi_{k_{n+1}} u_{n+1}) \right\|_{L^2(X)} \quad &= \quad \left\| \Pi_{k_{n+2}} \left[\prod_{j=1}^{n+1} \Pi_{k_j} \Lambda_m^{-1} \chi(\varepsilon^\delta \Lambda_m) u_j \right] \right\|_{L^2(X)} \\ &\lesssim_{n,X,N} \quad \frac{(k_3^\star)^{\nu+N}}{(k_1^\star - k_2^\star + k_3^\star)^N} \prod_{j=1}^{n+1} \left\| \Lambda_m^{-1} \chi(\varepsilon^\delta \Lambda_m) \Pi_{k_j} u_j \right\|_{L^2(X)}.\end{aligned}$$

We now remember that I_{k_j} is included in $[3\alpha(k_j - 1), 3\alpha(k_j + 1)]$ (see (11)) and that the cut-off function χ has support in $[-2, 2]$. If we therefore assume $k_j \geq 2$ and use the inequality $k_j - 1 \geq \frac{1}{3}(k_j + 1)$ then we have

$$\begin{aligned}\left\| \Lambda_m^{-1} \chi(\varepsilon^\delta \Lambda_m) \Pi_{k_j} u_j \right\|_{L^2(X)} \quad &\leq \quad C(m) \frac{\mathbf{1}_{[0,2\varepsilon^{-\delta}]} \left(\sqrt{m^2 + 9\alpha^2 (k_j - 1)^2)} \right)}{1 + k_j} \left\| \Pi_{k_j} u_j \right\|_{L^2(X)} \\ &\leq \quad C(m) \frac{\mathbf{1}_{[0,2\varepsilon^{-\delta}]}(3\alpha(k_j - 1))}{1 + k_j} \left\| \Pi_{k_j} u_j \right\|_{L^2(X)} \\ &\leq \quad C(m) \frac{\mathbf{1}_{[0,2\varepsilon^{-\delta}]}(\alpha(k_j + 1))}{1 + k_j} \left\| \Pi_{k_j} u_j \right\|_{L^2(X)}.\end{aligned}$$

Let us now explain why a similar inequality also holds true for $k_j \in \{0, 1\}$. Since χ is a cut-off function, it is clear that $\Lambda_m^{-1} \chi(\varepsilon^\delta \Lambda_m)$ is a linear bounded operator on $L^2(X)$ whose norm is less or equal to

$$C'(m) := \|\chi\|_{L^\infty(X)} \left\| \Lambda_m^{-1} \right\|_{L^2(X) \to L^2(X)}.$$

Noticing the inequality $2\varepsilon^{-\delta} \geq 2$, we get for any $k_j \in \{0, 1\}$:

$$\left\| \Lambda_m^{-1} \chi(\varepsilon^\delta \Lambda_m) \Pi_{k_j} u_j \right\|_{L^2(X)} \leq 2C'(m) \times \frac{\mathbf{1}_{[0,2\varepsilon^{-\delta}]}(k_j + 1)}{1 + k_j} \left\| \Pi_{k_j} u_j \right\|_{L^2(X)}.$$

Now introduce $\beta := \max\left(2, \frac{2}{\alpha}\right)$ so we have for any $k_j \in \mathbb{N}$:

$$\left\| \Lambda_m^{-1} \chi(\varepsilon^\delta \Lambda_m) \Pi_{k_j} u_j \right\|_{L^2(X)} \leq \max(C(m), 2C'(m)) \frac{\mathbf{1}_{[0,\beta\varepsilon^{-\delta}]}(k_j + 1)}{1 + k_j} \left\| \Pi_{k_j} u_j \right\|_{L^2(X)}.$$

Combining all the previous arguments, we have obtained a bound of the norm of the multilinear operator

$$\begin{array}{rcl} E_{k_1} \times \cdots \times E_{k_{n+1}} & \to & E_{k_{n+2}} \\ (u_1, \dots, u_{n+1}) & \mapsto & \Pi_{k_{n+2}} M_{m,n,\varepsilon,\delta}(u_1, \dots, u_{n+1}). \end{array}$$

Using Proposition 10 and (26), we may increase ν and get the following upper bound

$$\begin{aligned}&\left\| \left[\Pi_{k_{n+2}} \mathcal{L}_p^{-1}(M_{m,n,\varepsilon,\delta})\right] (\Pi_{k_1} u_1, \dots, \Pi_{k_{n+1}} u_{n+1}) \right\|_{L^2(X)} \\ &\quad \leq C(m,n,X,N) \frac{(k_1^\star)^{N_0 + \frac{d-1}{2}} (k_3^\star)^{\nu+N}}{(k_1^\star - k_2^\star + k_3^\star)^N} \prod_{j=1}^{n+1} \frac{\mathbf{1}_{[0,\beta\varepsilon^{-\delta}]}(1+k_j)}{1+k_j} \left\| \Pi_{k_j} u_j \right\|_{L^2(X)}.\end{aligned}$$

□

We now make the following decomposition

$$\begin{aligned}
\mathcal{L}_p^{-1}(M_{m,n,\varepsilon,\delta}) &= \mathcal{L}_{p,1}^{-1}(M_{m,n,\varepsilon,\delta}) + \mathcal{L}_{p,2}^{-1}(M_{m,n,\varepsilon,\delta}),\\
\mathcal{L}_{p,1}^{-1}(M_{m,n,\varepsilon,\delta})(u_1,\dots,u_{n+1}) &:= \sum_{\substack{k\in\mathbb{N}^{n+2}\setminus\mathcal{R}_{n+2,p}\\ k_1^\star\le 2k_2^\star}} \left[\Pi_{k_{n+2}}\mathcal{L}_p^{-1}(M_{m,n,\varepsilon,\delta})\right](\Pi_{k_1}u_1,\dots,\Pi_{k_{n+1}}u_{n+1}), \quad (39)\\
\mathcal{L}_{p,2}^{-1}(M_{m,n,\varepsilon,\delta})(u_1,\dots,u_{n+1}) &:= \sum_{\substack{k\in\mathbb{N}^{n+2}\setminus\mathcal{R}_{n+2,p}\\ 2k_2^\star< k_1^\star}} \left[\Pi_{k_{n+2}}\mathcal{L}_p^{-1}(M_{m,n,\varepsilon,\delta})\right](\Pi_{k_1}u_1,\dots,\Pi_{k_{n+1}}u_{n+1}). \quad (40)
\end{aligned}$$

In other words $\mathcal{L}_{p,1}^{-1}(M_{m,n,\varepsilon,\delta})$ is the part of $\mathcal{L}_p^{-1}(M_{m,n,\varepsilon,\delta})$ for which the largest two eigenvalues are of the same order whereas $\mathcal{L}_{p,2}^{-1}(M_{m,n,\varepsilon,\delta})$ is the part for which the largest eigenvalue is much larger than the other ones. Let us begin with the operator $\mathcal{L}_{p,1}^{-1}(M_{m,n,\varepsilon,\delta})$.

Proposition 16. — *There is $s_0 = s_0(d,n) > 0$ such that for any $s\in(s_0,+\infty)$, any $\varepsilon\in(0,1)$, any $\delta>0$, the series* (39) *which defines the operator $\mathcal{L}_{p,1}^{-1}(M_{m,n,\varepsilon,\delta})$ converges in the Banach space of bounded $(n+1)$-multilinear operators from $H^s(X)^{n+1}$ to $H^s(X)$. Moreover, for any $(u_1,\dots,u_{n+1})\in H^s(X)^{n+1}$, we have*

$$\left\|\mathcal{L}_{p,1}^{-1}(M_{m,n,\varepsilon,\delta})(u_1,\dots,u_{n+1})\right\|_{H^s(X)} \le C(m,n,X,s)\varepsilon^{-\delta(N_0+\frac{d-1}{2})}\prod_{j=1}^{n+1}\|u_j\|_{H^s(X)}.$$

Proof. We assume $\|u_j\|_{H^s(X)}=1$ for simplicity for each integer j. So we have the following relation with respect to the variable $(k_1,\dots,k_{n+1})\in\mathbb{N}^{n+1}$

$$\left\|\left(\prod_{j=1}^{n+1}(1+k_j)^s\left\|\Pi_{k_j}u_j\right\|_{L^2(X)}\right)\right\|_{\ell^2(\mathbb{N}^{n+1})} = 1. \quad (41)$$

We have to work with the following equality

$$\begin{aligned}
&\left\|\mathcal{L}_{p,1}^{-1}(M_{m,n,\varepsilon,\delta})(u_1,\dots,u_{n+1})\right\|_{H^s}^2\\
&\quad= \sum_{k_{n+2}\in\mathbb{N}}(1+k_{n+2})^{2s}\left\|\sum_{\substack{(k_1,\dots,k_{n+1})\in\mathbb{N}^{n+1}\\ k\in\mathbb{N}^{n+2}\setminus\mathcal{R}_{n+2,p};\ k_1^\star\le 2k_2^\star}}[\Pi_{k_{n+2}}\mathcal{L}_p^{-1}(M_{m,n,\varepsilon,\delta})](\Pi_{k_1}u_1,\dots,\Pi_{k_{n+1}}u_{n+1})\right\|_{L^2(X)}^2\\
&\le \sum_{k_{n+2}\in\mathbb{N}}(1+k_{n+2})^{2s}\left(\sum_{\substack{(k_1,\dots,k_{n+1})\in\mathbb{N}^{n+1}\\ k\in\mathbb{N}^{n+2}\setminus\mathcal{R}_{n+2,p};\ k_1^\star\le 2k_2^\star}}\left\|[\Pi_{k_{n+2}}\mathcal{L}_p^{-1}(M_{m,n,\varepsilon,\delta})](\Pi_{k_1}u_1,\dots,\Pi_{k_{n+1}}u_{n+1})\right\|_{L^2(X)}\right)^2.
\end{aligned}$$

Corollary 15 with $N=0$ gives the following upper bounds

$$\begin{aligned}
&\left\|[\Pi_{k_{n+2}}\mathcal{L}_p^{-1}(M_{m,n,\varepsilon,\delta})](\Pi_{k_1}u_1,\dots,\Pi_{k_{n+1}}u_{n+1})\right\|_{L^2(X)}\\
&\qquad\le C(m,n,X)(k_1^\star)^{N_0+\frac{d-1}{2}}(k_3^\star)^{\nu}\mathbf{1}_{[0,\beta\varepsilon^{-\delta}]}(1+\max(k_1,\dots,k_{n+1}))\prod_{j=1}^{n+1}\frac{\left\|\Pi_{k_j}u_j\right\|_{L^2(X)}}{(1+k_j)}\\
&\qquad\le C(m,n,X)\frac{(k_1^\star)^{N_0+\frac{d-1}{2}}(k_3^\star)^{\nu}\mathbf{1}_{[0,\beta\varepsilon^{-\delta}]}(1+\max(k_1,\dots,k_{n+1}))}{\prod_{j=1}^{n+1}(1+k_j)^{s+1}}\prod_{j=1}^{n+1}(1+k_j)^s\left\|\Pi_{k_j}u_j\right\|_{L^2(X)}.
\end{aligned}$$

Thanks to the Cauchy-Schwarz inequality with (41) and a symmetry argument, we get the following bounds

$$\begin{aligned}
&\left\|\mathcal{L}_{p,1}^{-1}(M_{m,n,\varepsilon,\delta})(u_1,\dots,u_{n+1})\right\|_{H^s}^2\\
&\lesssim_{m,nX} \sum_{k_{n+2}\in\mathbb{N}} (1+k_{n+2})^{2s}\left(\sum_{\substack{(k_1,\dots,k_{n+1})\in\mathbb{N}^{n+1}\\ k_1^\star\le 2k_2^\star}}\left(\frac{(k_1^\star)^{N_0+\frac{d-1}{2}}(k_3^\star)^{\nu}\mathbf{1}_{[0,\beta\varepsilon^{-\delta}]}(1+\max(k_1,\dots,k_{n+1}))}{\prod_{j=1}^{n+1}(1+k_j)^{s+1}}\right)^2\right)\\
&\lesssim_{m,n,X} \sum_{\substack{k\in\mathbb{N}^{n+2}\\ k_1^\star\le 2k_2^\star}} (1+k_{n+2})^{2s}(k_1^\star)^{2N_0+d-1}(k_3^\star)^{2\nu}\mathbf{1}_{[0,\beta\varepsilon^{-\delta}]}(1+\max(k_1,\dots,k_{n+1}))\prod_{j=1}^{n+1}\frac{1}{(1+k_j)^{2s+2}}\\
&\lesssim_{m,n,X} \sum_{\substack{k\in\mathbb{N}^{n+2}\\ k_1^\star\le 2k_2^\star\\ k_1\le\cdots\le k_{n+1}}} (1+k_{n+2})^{2s}(k_1^\star)^{2N_0+d-1}(k_3^\star)^{2\nu}\mathbf{1}_{[0,\beta\varepsilon^{-\delta}]}(1+k_{n+1})\prod_{j=1}^{n+1}\frac{1}{(1+k_j)^{2s+2}}.
\end{aligned}$$

Note that the conditions $k_1\le\cdots\le k_{n+1}$ imply the following two inequalities $k_2^\star\le k_{n+1}$ and $k_3^\star\le k_n$. The condition $k_1^\star\le 2k_2^\star\le 2k_{n+1}$ leads to the following upper bound

$$C(m,n,X)\sum_{\substack{k\in\mathbb{N}^{n+2}\\ k_1^\star\le 2k_2^\star\\ k_1\le\cdots\le k_{n+1}}}\frac{(1+k_{n+2})^{2s}k_{n+1}^{2N_0+d-1}\mathbf{1}_{[0,\beta\varepsilon^{-\delta}]}(1+k_{n+1})}{(1+k_n)^{2s+2-2\nu}(1+k_{n+1})^{2s+2}}\prod_{j=1}^{n-1}\frac{1}{(1+k_j)^{2s+2}}. \tag{42}$$

We claim that the inequality $k_{n+2}\le 2k_{n+1}$ always holds true. This is indeed obvious if $k_{n+2}\le k_{n+1}$ holds true. If k_{n+2} is greater than k_{n+1}, then the condition $k_1^\star\le 2k_2^\star$ means exactly that $k_{n+2}\le 2k_{n+1}$ holds true. This allows us to get rid of s and k_{n+2} :

$$\frac{(1+k_{n+2})^{2s}k_{n+1}^{2N_0+d-1}\mathbf{1}_{[0,\beta\varepsilon^{-\delta}]}(1+k_{n+1})}{(1+k_{n+1})^{2s+2}}\le C(s)(1+k_{n+1})^{2N_0+d-3}\mathbf{1}_{[0,\beta\varepsilon^{-\delta}]\times[0,2\beta\varepsilon^{-\delta}]}(1+k_{n+1},1+k_{n+2}).$$

We now easily see that if $2s+2$ is large enough then we can bound (42) by

$$C(m,n,X,s)\left(\sum_{k_{n+1}\in\mathbb{N}}(1+k_{n+1})^{2N_0+d-3}\mathbf{1}_{[0,\beta\varepsilon^{-\delta}]}(1+k_{n+1})\right)\left(\sum_{k_{n+2}\in\mathbb{N}}\mathbf{1}_{[0,2\beta\varepsilon^{-\delta}]}(1+k_{n+2})\right),$$

which is less than $C(m,n,X,s)\varepsilon^{-\delta(2N_0+d-1)}$. □

Let us now study the operator $\mathcal{L}_{p,2}^{-1}(M_{m,n,\varepsilon,\delta})$.

Proposition 17. — *There is $s_0=s_0(d,n,m)>0$ such that for any $s\in(s_0,+\infty)$, any $\varepsilon\in(0,1)$ and any $\delta>0$, the series (40) which defines the operator $\mathcal{L}_{p,2}^{-1}(M_{m,n,\varepsilon,\delta})$ converges in the Banach space of bounded $(n+1)$-multilinear operators from $H^s(X)^{n+1}$ to $H^s(X)$. Moreover, for any $(u_1,\dots,u_{n+1})\in H^s(X)^{n+1}$, we have*

$$\left\|\mathcal{L}_{p,2}^{-1}(M_{m,n,\varepsilon,\delta})(u_1,\dots,u_{n+1})\right\|_{H^s(X)}\le C(m,n,X,s)\prod_{j=1}^{n+1}\|u_j\|_{H^s(X)}.$$

Proof. We begin as in the proof of Proposition 16 by assuming that each u_j has a norm 1 in $H^s(X)$. A symmetry argument and Corollary 15 give, for any integer N, the following upper bound

$$\begin{aligned}
&\left\|\mathcal{L}_{p,2}^{-1}(M_{m,n,\varepsilon,\delta})(u_1,\dots,u_{n+1})\right\|_{H^s}^2\\
&\le C(m,n,X,N)\sum_{\substack{k\in\mathbb{N}^{n+2}\\ k_1\le\cdots\le k_{n+1}\\ 2k_2^\star<k_1^\star}}\frac{(1+k_{n+2})^{2s}(k_1^\star)^{2N_0+d-1}(k_3^\star)^{2\nu+2N}}{(k_1^\star-k_2^\star+k_3^\star)^{2N}}\prod_{j=1}^{n+1}\frac{1}{(1+k_j)^{2s+2}}.
\end{aligned}$$

In the previous sum, the inequality $k_1^\star - k_2^\star + k_3^\star \geq \frac{1}{2}k_1^\star$ trivially holds. Hence, one gets the bound

$$C(m,n,X,N) \sum_{\substack{k\in\mathbb{N}^{n+2}\\ k_1\leq\cdots\leq k_{n+1}}} \frac{(1+k_{n+2})^{2s}(k_3^\star)^{2\nu+2N}}{(k_1^\star)^{2N-2N_0-d+1}} \prod_{j=1}^{n+1} \frac{1}{(1+k_j)^{2s+2}}. \tag{43}$$

We consider below three cases according to the position of k_{n+2} with respect to k_n and k_{n+1}. In the computations below, we first choose s large enough and then N large enough such that all the computations are licit.

Case $k_n \leq k_{n+1} \leq k_{n+2}$**.** Hence, we have $k_1^\star = 1+k_{n+2}$ and $k_3^\star = 1+k_n$. Provided that $2s+2>1$, we can write

$$\begin{aligned}
&\sum_{\substack{k\in\mathbb{N}^{n+2}\\ k_1\leq\cdots\leq k_{n+2}}} \frac{(1+k_{n+2})^{2s}(1+k_n)^{2N+2\nu}}{(1+k_{n+2})^{2N-2N_0-d+1}} \prod_{j=1}^{n+1} \frac{1}{(1+k_j)^{2s+2}}\\
&\qquad \leq C(n,s,X) \sum_{\substack{(k_n,k_{n+1},k_{n+2})\in\mathbb{N}^3\\ k_n\leq k_{n+1}\leq k_{n+2}}} \frac{(1+k_n)^{2N-2s+2\nu-2}}{(1+k_{n+2})^{2N-2s-2N_0-d+1}(1+k_{n+1})^{2s+2}}.
\end{aligned}$$

We may assume that $2N-2s+2\nu-2$ is positive and hence get the following upper bound

$$\sum_{\substack{(k_{n+1},k_{n+2})\in\mathbb{N}^2\\ k_{n+1}\leq k_{n+2}}} \frac{(1+k_{n+1})^{2N-2s+2\nu-1}}{(1+k_{n+2})^{2N-2s-2N_0-d+1}(1+k_{n+1})^{2s+2}} = \sum_{\substack{(k_{n+1},k_{n+2})\in\mathbb{N}^2\\ k_{n+1}\leq k_{n+2}}} \frac{(1+k_{n+1})^{2N-4s+2\nu-3}}{(1+k_{n+2})^{2N-2s-2N_0-d+1}}.$$

With the same idea, we assume that the exponent $2N-4s+2\nu-3$ is positive in order to get the bound

$$\sum_{k_{n+2}\in\mathbb{N}} \frac{(1+k_{n+2})^{2N-4s+2\nu-2}}{(1+k_{n+2})^{2N-2s-2N_0-d+1}} = \sum_{k_{n+2}\in\mathbb{N}} \frac{1}{(1+k_{n+2})^{2s-2\nu-2N_0-d+3}}.$$

The previous series converges if we choose s large enough (this condition is independent of N).

Case $k_n \leq k_{n+2} \leq k_{n+1}$**.** Hence, we have $k_1^\star = 1+k_{n+1}$ and $k_3^\star = 1+k_n$. The bound (43) becomes up to a multiplicative constant $C(m,n,X,N)$:

$$\sum_{\substack{k\in\mathbb{N}^{n+2}\\ k_1\leq\cdots\leq k_n\leq k_{n+2}\leq k_{n+1}}} \frac{(1+k_{n+2})^{2s}}{(1+k_n)^{2s-2N-2\nu+2}(1+k_{n+1})^{2s+2N-2N_0-d+3}} \prod_{j=1}^{n-1} \frac{1}{(1+k_j)^{2s+2}}. \tag{44}$$

We follow the same strategy of that of the previous case. We first choose s large enough and then N large enough. Hence, we can bound (44) by

$$\begin{aligned}
&C(n,s,X) \sum_{\substack{(k_n,k_{n+1},k_{n+2})\in\mathbb{N}^3\\ k_n\leq k_{n+2}\leq k_{n+1}}} \frac{(1+k_{n+2})^{2s}(1+k_n)^{-2s+2N+2\nu-2}}{(1+k_{n+1})^{2s+2N-2N_0-d+3}}\\
&\qquad \leq C(n,s,X) \sum_{\substack{(k_{n+1},k_{n+2})\in\mathbb{N}^2\\ k_{n+2}\leq k_{n+1}}} \frac{(1+k_{n+2})^{2N+2\nu-1}}{(1+k_{n+1})^{2s+2N-2N_0-d+3}}\\
&\qquad \leq C(n,x,X) \sum_{k_{n+1}\in\mathbb{N}} \frac{(1+k_{n+1})^{2N+2\nu}}{(1+k_{n+1})^{2s+2N-2N_0-d+3}}\\
&\qquad \leq C(n,s,X) \sum_{k_{n+1}\in\mathbb{N}} \frac{1}{(1+k_{n+1})^{2s-2\nu-2N_0-d+3}} < +\infty.
\end{aligned}$$

Case $k_{n+2} \leq k_n \leq k_{n+1}$. For this case, we have $k_1^\star = 1 + k_{n+1}$. In the case $n = 1$, we have $k_3^\star = 1 + k_3$ and the series in (43) is less than

$$\sum_{\substack{k\in\mathbb{N}^3\\ k_3\leq k_1\leq k_2}} \frac{(1+k_3)^{2N+2s+2\nu}}{(1+k_2)^{2N+2s-2N_0-d+3}(1+k_1)^{2s+2}} \lesssim \sum_{\substack{k\in\mathbb{N}^2\\ k_1\leq k_2}} \frac{(1+k_1)^{2N+2\nu-1}}{(1+k_2)^{2N+2s-2N_0-d+3}}$$
$$\lesssim \sum_{k_2\in\mathbb{N}} \frac{1}{(1+k_2)^{2s-2N_0-d+3-2\nu}} < +\infty.$$

In the case $n \geq 2$, we have $k_3^\star = 1 + \max(k_{n+2}, k_{n-1})$ and the bound (43) becomes :

$$\sum_{\substack{k\in\mathbb{N}^{n+2}\\ k_1\leq\cdots\leq k_{n+1}\\ k_{n+2}\leq k_n}} \frac{(1+k_{n+2})^{2s}(1+\max(k_{n+2},k_{n-1}))^{2N+2\nu}}{(1+k_{n+1})^{2N-2N_0-d+1}} \prod_{j=1}^{n+1} \frac{1}{(1+k_j)^{2s+2}}.$$

As in the previous two cases, one chooses $2s + 2$ to be greater than 1 so that we are reduced to bound

$$C(n,s,X) \sum_{\substack{(k_{n-1},k_n,k_{n+1},k_{n+2})\in\mathbb{N}^4\\ k_{n-1}\leq k_n\leq k_{n+1}\\ k_{n+2}\leq k_n}} \frac{(1+k_{n+2})^{2s}(1+\max(k_{n+2},k_{n-1}))^{2N+2\nu}}{(1+k_{n+1})^{2s+2N-2N_0-d+3}(1+k_{n-1})^{2s+2}(1+k_n)^{2s+2}}. \tag{45}$$

We now eliminate k_n in the previous sum by introducing the condition $k_{n+2} \leq k_{n+1}$ and summing in $k_n \in [k_{n+2}, +\infty[\cap\mathbb{N}$. This allows us to bound (45) by

$$C(n,s,X) \sum_{\substack{(k_{n-1},k_{n+1},k_{n+2})\in\mathbb{N}^3\\ k_{n-1}\leq k_{n+1}\\ k_{n+2}\leq k_{n+1}}} \frac{(1+\max(k_{n+2},k_{n-1}))^{2N+2\nu}}{(1+k_{n+1})^{2s+2N-2N_0-d+3}(1+k_{n-1})^{2s+2}(1+k_{n+2})}. \tag{46}$$

Using the same ideas as above, we can get the following estimates of (46) if one chooses s large enough and then N large enough

$$\sum_{\substack{(k_{n-1},k_{n+1},k_{n+2})\in\mathbb{N}^3\\ k_{n-1}\leq k_{n+1}\\ k_{n+2}\leq k_{n+1}}} \frac{(1+k_{n+2})^{2N+2\nu-1}}{(1+k_{n+1})^{2s+2N-2N_0-d+3}(1+k_{n-1})^{2s+2}} + \frac{(1+k_{n-1})^{2N-2s+2\nu-2}}{(1+k_{n+1})^{2s+2N-2N_0-d+3}(1+k_{n+2})}$$
$$\lesssim \sum_{\substack{(k_{n+1},k_{n+2})\in\mathbb{N}^2\\ k_{n+2}\leq k_{n+1}}} \frac{(1+k_{n+2})^{2N+2\nu-1}}{(1+k_{n+1})^{2N+2s-2N_0-d+3}} + \frac{1}{(1+k_{n+1})^{4s-2N_0-d+4-2\nu}(1+k_{n+2})}$$
$$\lesssim \sum_{k_{n+1}\in\mathbb{N}} \frac{1}{(1+k_{n+1})^{2s-2N_0-d+3-2\nu}} + \frac{\ln(1+k_{n+1})}{(1+k_{n+1})^{4s-2N_0-d+4-2\nu}} < +\infty.$$

□

8. Simultaneous approximation to algebraic numbers

We need a deep result about simultaneous Diophantine approximations proved by Schmidt ([**Sch80**, Corollary 1E, page 152] or [**Sch70**, Corollary of Theorem 2]).

***Theorem 18* (Schmidt).** — *Suppose $\alpha_1, \ldots, \alpha_\ell$ are ℓ real algebraic numbers such that $1, \alpha_1, \ldots, \alpha_\ell$ are linearly independent over the field $\mathbb{Q}$ and consider $\eta > 0$. Then there are only finitely many $(\ell+1)$-tuples $(q_1, \ldots, q_\ell, p) \in \mathbb{Z}^{\ell+1}$ such that*

$$\max(|q_1|,\ldots,|q_\ell|) > 0 \qquad \textit{and} \qquad |\alpha_1 q_1 + \cdots + \alpha_\ell q_\ell - p| < \frac{1}{\max(|q_1|,\ldots,|q_\ell|)^{\ell+\eta}}. \tag{47}$$

The previous statement is indeed equivalent to the following one.

Theorem 19. — *Suppose $\alpha_1, \dots, \alpha_\ell$ are ℓ real algebraic numbers such that $1, \alpha_1, \dots, \alpha_\ell$ are linearly independent over the field $\mathbb{Q}$ and consider $\eta > 0$. There exists a constant $C > 0$ which depends on $(\alpha_1, \dots, \alpha_\ell, \ell, \eta)$ such that for any $(\ell+1)$-tuples $(p, q_1, \dots, q_\ell) \in \mathbb{Z}^{\ell+1} \backslash \{0\}$, we have*

$$|\alpha_1 q_1 + \cdots + \alpha_\ell q_\ell - p| \geq \frac{C}{\max(|p|, |q_1|, \dots, |q_\ell|)^{\ell+\eta}}.$$

Proof of the implication "Theorem 18 ⇒ Theorem 19". The map

$$\aleph : (p, q_1, \dots, q_\ell) \mapsto |\alpha_1 q_1 + \cdots + \alpha_\ell q_\ell - p| \times \max(|q_1|, \dots, |q_\ell|)^{\ell+\eta}$$

sends $\mathbb{Z} \times \mathbb{Z}^\ell \backslash \{0\}$ to $(0, +\infty)$ and takes only finitely many values below 1. The map $\aleph$ is thus bounded from below by a constant which belongs to $(0,1)$. This fact proves Theorem 19 if $(p, q_1, \dots, q_\ell)$ belongs to $\mathbb{Z} \times \mathbb{Z}^\ell \backslash \{0\}$. The case $(p, 0, \dots, 0) \in \mathbb{Z}\backslash\{0\} \times \{0\}^\ell$ is obvious since $|p| \geq 1$.

Proof of the implication "Theorem 19 ⇒ Theorem 18". Let us consider $(q_1, \dots, q_\ell, p) \in \mathbb{Z}^{\ell+1} \backslash \{0\}$ such that (47) holds true. This clearly implies the inequality $|p| \lesssim \max(|q_1|, \dots, |q_\ell|)$ and hence $\max(|q_1|, \dots, |q_\ell|) \simeq \max(|p|, |q_1|, \dots, |q_\ell|)$. Thanks to (47) and Theorem 19 with $\eta/2$, we get

$$\begin{aligned} |\alpha_1 q_1 + \cdots + \alpha_\ell q_\ell - p| &\geq \frac{C}{\max(|q_1|, \dots, |q_\ell|)^{\ell+\frac{\eta}{2}}} \\ \frac{1}{\max(|q_1|, \dots, |q_\ell|)^{\ell+\eta}} &\geq \frac{C}{\max(|q_1|, \dots, |q_\ell|)^{\ell+\frac{\eta}{2}}} \\ C^{-\frac{2}{\eta}} &\geq \max(|q_1|, \dots, |q_\ell|). \end{aligned}$$

There are only many finitely ℓ-tuples $(q_1, \dots, q_\ell)$ which satisfy the latter inequality. To finish the proof, we notice that p also runs over a finite set because of the estimate $|p| \lesssim \max(|q_1|, \dots, |q_\ell|)$.

This achieves the equivalence of Theorem 18 and Theorem 19. Let us now recall that the degree of an algebraic number $\alpha \in \mathbb{C}$ is defined by the formula

$$\deg(\alpha) := \min\{\ell \in \mathbb{N}^\star, \quad \exists P \in \mathbb{Q}[X]\backslash\{0\} \quad P(\alpha) = 0 \quad \text{and} \quad \deg P = \ell\}.$$

In the above definition, we may assume that the coefficients of P are integers. Note that if $\deg(\alpha) > \ell$ holds then $|P(\alpha)| > 0$ for any polynomial $P \in \mathbb{Z}[X]\backslash\{0\}$ of degree less or equal to ℓ. For our purpose, we need to quantify how small can be $|P(\alpha)|$.

Corollary 20. — *Consider an integer $\ell \geq 1$ and a real algebraic number α of degree $\deg(\alpha) > \ell$. For any $\eta > 0$, there is a constant $C(\eta, \ell, \alpha) > 0$ such that for any $(q_0, \dots, q_\ell) \in \mathbb{Z}^{\ell+1} \backslash \{0\}$ we have*

$$\left|\sum_{k=0}^{\ell} q_k \alpha^k\right| \geq \frac{C(\eta, \ell, \alpha)}{\max(|q_0|, \dots, |q_\ell|)^{\ell+\eta}}. \tag{48}$$

The following result ([**Bug04**, page 75, Theorem 4.2], [**Spr69**, see page 1 and page 63]) is the analytic counterpart of the previous result.

***Theorem 21* (Sprindžuk)**. — *Almost every real number α (in the sense of Lebesgue) satisfies the following : for any integer $\ell \geq 1$, for any $\eta > 0$, there is a constant $C(\eta, \ell, \alpha) > 0$ such that for any $(q_0, \dots, q_\ell) \in \mathbb{Z}^{\ell+1} \backslash \{0\}$ we have*

$$\left|\sum_{k=0}^{\ell} q_k \alpha^k\right| \geq \frac{C(\eta, \ell, \alpha)}{\max(|q_0|, \dots, |q_\ell|)^{\ell+\eta}}. \tag{49}$$

Remark 22. — *Theorem 21 is often written under another form in the literature. For any real number α and any integer $\ell \geq 1$, let us denote by $w_\ell(\alpha)$ the upper bound of the real numbers w for which there exist infinitely many polynomials $P = q_0 + \cdots + q_\ell X^\ell \in \mathbb{Z}[X]$ such that*

$$0 < |P(\alpha)| < \frac{1}{\max(|q_0|, \dots, |q_\ell|)^w}.$$

The Sprindžuk theorem is usually stated as follows : $w_\ell(\alpha) = \ell$ for almost every $\alpha \in \mathbb{R}$. Using similar arguments of those of the above equivalence of Theorem 18 and Theorem 19, we easily see that the formulation of Theorem 21 is nothing else than the inequality $w_\ell(\alpha) \leq \ell$.

Remark 23. — *The estimates* (48) *and* (49) *are relevant if and only if* $|\sum_{k=0}^{\ell} q_k\alpha^k|$ *is less than* 1*, we may therefore assume that the largest two integers among* $q_0,\dots,q_\ell$ *are of the same order.*

We now prove Proposition 9. The reader will easily check that if one uses Theorem 21 instead of Corollary 20 in the following proof, then one obtains an alternative proof of the Delort-Szeftel estimates (Proposition 7) in the particular case $\{\lambda_k^2, k\in\mathbb{N}\}\subset\mathbb{N}$. The condition $\deg(m)>2^{n+1}$ implies that the degree of the algebraic number $\mu := m^2$ satisfies

$$\deg(\mu)\geq \frac{1}{2}\deg(m) > 2^n. \tag{50}$$

Case $n = 1$. For the convenience of the reader, we sketch the idea for $n = 1$. The following eight numbers

$$\pm\sqrt{\mu+k_1}\pm\sqrt{\mu+k_2}\pm\sqrt{\mu+k_3}. \tag{51}$$

are the roots of a polynomial of the following form

$$X^8+v_6(\mu,k_1,k_2,k_3)X^6+v_4(\mu,k_1,k_2,k_3)X^4+v_2(\mu,k_1,k_2,k_3)X^2+v_0(\mu,k_1,k_2,k_3),$$

where v_6, v_4, v_2 and v_0 are functions which only depend on (μ,k_1,k_2,k_3). Moreover, it is classical that the modulus of any root of the previous polynomial in X is greater or equal to

$$\min\left(1,\frac{\sqrt{|v_0(\mu,k_1,k_2,k_3)|}}{\sqrt{1+|v_6(\mu,k_1,k_2,k_3)|+|v_4(\mu,k_1,k_2,k_3)|+|v_2(\mu,k_1,k_2,k_3)|}}\right). \tag{52}$$

Note now that the functions v_2, v_4 and v_6 are homogeneous polynomials with respect to (μ,k_1,k_2,k_3) whose degrees are less or equal to 3. This leads us to bound from below the second term in (52) by

$$C\frac{\sqrt{|v_0(\mu,k_1,k_2,k_3)|}}{(\mu+k_1+k_2+k_3)^{3/2}}\geq C(m)\frac{\sqrt{|v_0(\mu,k_1,k_2,k_3)|}}{(1+\max(k_1,k_2,k_3))^{3/2}}.$$

As $v_0(\mu,k_1,k_2,k_3)$ is the product of the eight roots (51), a straightforward computation gives

$$v_0(\mu,k_1,k_2,k_3)=[3\mu^2+2(k_1+k_2+k_3)\mu+2(k_1k_2+k_1k_3+k_2k_3)-k_1^2-k_2^2-k_3^2]^2.$$

We now understand why the Schmidt theorem is useful. As we assumed that μ is algebraic of degree larger than 2, Corollary 20 and Remark 23 allows us to bound from below $|v_0(\mu,k_1,k_2,k_3)|$ by a negative power of the second largest integer among

$$3,\quad k_1+k_2+k_3,\quad 2(k_1k_2+k_1k_3+k_2k_3)-k_1^2-k_2^2-k_3^2.$$

More precisely, we get for any $\eta>0$ and $N_0>7$ the following lower bounds

$$\begin{aligned}|v_0(\mu,k_1,k_2,k_3)| &\geq \frac{C(m,\eta)}{(1+k_1+k_2+k_3)^{4+\eta}}\\ |\pm\sqrt{\mu+k_1}\pm\sqrt{\mu+k_2}\pm\sqrt{\mu+k_3}| &\geq \frac{C(m,N_0)}{(1+k_1+k_2+k_3)^{\frac{1}{2}N_0}}.\end{aligned}$$

Case $n\geq 2$. One can use the same argument if $n\geq 3$ is odd (see remark 25). If n is even, then one has to modify the proof because of the resonant terms. We prefer to give a unified proof of the case $n\geq 2$ whatever the parity of n is. We have to prove (14) in the nonresonant regime. Hence, there are nonnegative integers $K_1<\cdots<K_\ell$ and coefficients $\rho_1,\dots,\rho_\ell\in\mathbb{Z}^\star$ such that

$$\sum_{j=1}^{p}\sqrt{\mu+k_k}-\sum_{j=p+1}^{n+2}\sqrt{\mu+k_j}=\rho_1\sqrt{\mu+K_1}+\cdots+\rho_\ell\sqrt{\mu+K_\ell},$$

$$1\leq\ell\leq n+2,\qquad 1\leq|\rho_1|+\cdots+|\rho_\ell|\leq n+2,\qquad K_\ell\leq\max(k_1,\dots,k_{n+2}). \tag{53}$$

The subcase $\ell=1$ presents no difficulty because of the trivial inequality $|\rho_1\sqrt{\mu+K_1}|\geq\sqrt{\mu}$ which implies (14). The subcase $\ell\geq 2$ needs the following result (that is proved below).

Proposition 24. — *Let us fix an integer* $\ell\geq 2$ *and a tuple* $(\rho_1,\dots,\rho_\ell)\in(\mathbb{Z}^\star)^\ell$*. There are polynomials* $Z_0,\dots,Z_{2^{\ell-2}}\in\mathbb{Z}[X_1,\dots,X_\ell]$*, such that*

i) the coefficients of the above polynomials merely depend on $(\ell,\rho_1,\dots,\rho_\ell)$*,*

ii) for any $j\in[0,2^{\ell-2}]\cap\mathbb{N}$ *one has* $\deg(Z_j)\leq 2^{\ell-2}-j$*,* *iii) for any any tuple* $(K_1,\dots,K_\ell)\in(0,+\infty)^\ell$ *of* ***distinct*** *numbers, the polynomial* $\mu\mapsto\sum_{j=0}^{2^{\ell-2}}Z_j(K_1,\dots,K_\ell)\mu^j$ *is not zero,*

iv) for any $\mu>0$ *there is* $C(\ell,\rho_1,\dots,\rho_\ell,\mu)>0$ *such that the following inequality holds true for any tuple* $(K_1,\dots,K_\ell)\in(0,+\infty)^\ell$

$$|\rho_1\sqrt{\mu+K_1}+\cdots+\rho_\ell\sqrt{\mu+K_\ell}|\geq\min\left(1,\frac{C(\ell,\rho_1,\dots,\rho_\ell,\mu)}{(1+\max(K_1,\dots,K_\ell)^{\frac12(2^{\ell-1}-1)}}\left|\sum_{j=0}^{2^{\ell-2}}Z_j(K_1,\dots,K_\ell)\mu^j\right|\right).\tag{54}$$

Let us consider the polynomials $Z_0,\dots,Z_{2^{\ell-2}}$ as in Proposition 24. Since the integer n is fixed, the conditions (53) imply that there are a finite number of choices of $(\ell,\rho_1,\dots,\rho_\ell)$ and a finite number of polynomials $Z_0,\dots,Z_{2^{\ell-2}}$. Remembering the equality $\mu=m^2$, one may bound from below the constant $C(\ell,\rho_1,\dots,\rho_\ell,\mu)$ by $C(n,m)>0$. For the same reason, one can bound all the coefficients of $Z_0,\dots,Z_{2^{\ell-2}}$ by an (ineffective) constant $C(n)>0$. Point ii) of Proposition 24 ensures that the second largest integer among

$$Z_0(K_1,\dots,K_\ell),\dots,Z_{2^{\ell-2}}(K_1,\dots,K_\ell)$$

is less than $C(n)(1+\max(K_1,\dots,K_\ell))^{2^{\ell-2}-1}$. Thanks to (50) and (53), one has $\deg(\mu)>2^n\geq2^{\ell-2}$. Note now that the polynomial $\mu\mapsto\sum_{j=0}^{2^{\ell-2}}Z_j(K_1,\dots,K_\ell)\mu^j$ is not zero (Point iii of Proposition 24). For any $\eta>0$, Corollary 20 and Remark 23 give us the following lower bound

$$\begin{aligned}\left|\sum_{j=0}^{2^{\ell-2}}Z_j(K_1,\dots,K_\ell)\mu^j\right| &\geq \frac{C(\eta,m,n)}{(1+\max(K_1,\dots,K_\ell))^{(2^{\ell-2}-1)(\eta+2^{\ell-2})}}\\ &\geq \frac{C(\eta,m,n)}{\left(1+\sqrt{\max(k_1,\dots,k_{n+2}}\right)^{2(2^{\ell-2}-1)(\eta+2^{\ell-2})}}.\end{aligned}$$

The inequality (54) allows us to conclude that any number $N_0>(2^{\ell-1}-1)+2(2^{\ell-2}-1)2^{\ell-2}=2^{2\ell-3}-1$ is convenient in (14) (the maximal value $2^{2n+1}-1$ is obtained for $\ell=n+2$).

9. Proof of Proposition 24

Let us introduce the polynomials

$$U(X,T_1,\dots,T_\ell)=\prod_{(\omega_1,\dots,\omega_\ell)=(\pm1,\dots,\pm1)}\left(X+\sum_{k=1}^{\ell}\omega_k\rho_kT_k\right)\in\mathbb{Z}[X,T_1,\dots,T_\ell],$$

$$\widetilde{U}(T_1,\dots,T_\ell)=\prod_{(\omega_2,\dots,\omega_\ell)=(\pm1,\dots,\pm1)}\left(\rho_1T_1+\sum_{k=2}^{\ell}\omega_k\rho_kT_k\right)\in\mathbb{Z}[T_1,\dots,T_\ell].$$

It is clear that U is even with respect to each variable and we moreover have a decomposition

$$U(X,T_1,\dots,T_\ell)=X^{2^\ell}+\sum_{i=0}^{2^{\ell-1}-1}V_i(T_1^2,\dots,T_\ell^2)X^{2i},$$

where $V_0,\dots,V_{2^{\ell-1}-1}$ are homogeneous polynomials satisfying $\deg(V_i)=2^{\ell-1}-i$ for each integer $i\in[0,2^{\ell-1}-1]$. It is moreover clear that the coefficients of each V_i merely depends on $(\ell,\rho_1,\dots,\rho_\ell)$.

The polynomial V_0 will play a substantial role and we thus need to remark the following

$$\begin{aligned}V_0(T_1^2,\dots,T_\ell^2) &= U(0,T_1,\dots,T_\ell)\\ &= \widetilde{U}(T_1,\dots,T_\ell)\widetilde{U}(-T_1,\dots,T_\ell)\\ &= \widetilde{U}(T_1,\dots,T_\ell)^2.\end{aligned}$$

As above, the $2^{\ell-1}$-homogeneous polynomial $\widetilde{U}(T_1,\dots,T_\ell)$ is even with respect to each variable (this fact uses the assumption $\ell\geq2$) and thus belongs to $\mathbb{Z}[T_1^2,\dots,T_\ell^2]$. Hence, there is $W\in\mathbb{Z}[T_1,\dots,T_\ell]$ such that

$$\widetilde{U}(T_1,\dots,T_\ell)=W(T_1^2,\dots,T_\ell^2),\quad\deg(W)=2^{\ell-2},$$

$$V_0(T_1^2,\dots,T_\ell^2)=W(T_1^2,\dots,T_\ell^2)^2. \tag{55}$$

Let us now denote by x a real number of the form

$$x:=\sum_{k=1}^{\ell}\rho_k\sqrt{\mu+K_k}.$$

We may begin the analysis of the lower bound of $|x|$. Let us write

$$0=U\left(x,\sqrt{\mu+K_1},\dots,\sqrt{\mu+K_\ell}\right)=x^{2^\ell}+\sum_{i=0}^{2^{\ell-1}-1}V_i(\mu+K_1,\dots,\mu+K_\ell)x^{2i}.$$

If $|x|\geq 1$ holds true, then (54) is obvious. If $|x|<1$ holds true then one can write

$$\begin{aligned}|V_0(\mu+K_1,\dots,\mu+K_\ell)| &\leq |x|^2\left(1+\sum_{i=1}^{2^{\ell-1}-1}|V_i(\mu+K_1,\dots,\mu+K_\ell)|\right)\\ &\leq C(\ell,\rho_1,\dots,\rho_\ell)|x|^2\left(1+\sum_{i=1}^{2^{\ell-1}-1}(\mu+K_1+\dots+K_\ell)^{2^{\ell-1}-i}\right)\\ &\leq C(\ell,\rho_1,\dots,\rho_\ell,\mu)|x|^2\left(1+\max(K_1,\dots,K_\ell)\right)^{2^{\ell-1}-1}.\end{aligned}$$

Then (55) gives us

$$|x|\geq C(\ell,\rho_1,\dots,\rho_\ell,\mu)\frac{|W(\mu+K_1,\dots,\mu+K_\ell)|}{(1+\max(K_1,\dots,K_\ell))^{\frac{1}{2}(2^{\ell-1}-1)}}. \tag{56}$$

By decomposing W, we can write

$$W(\mu+K_1,\dots,\mu+K_\ell)=\sum_{i_1+\dots+i_\ell=2^{\ell-2}}c(i_1,\dots,i_\ell)(\mu+K_1)^{i_1}\dots(\mu+K_\ell)^{i_\ell},\qquad c(i_1,\dots,i_\ell)\in\mathbb{Z}.$$

Let us admit for a moment that $\mu\mapsto W(\mu+K_1,\dots,\mu+K_\ell)$ is a nonzero polynomial (with respect to μ) (see below). One can now introduce the polynomials Z_j of the statement of Proposition 24 in the following way :

$$W(\mu+K_1,\dots,\mu+K_\ell)=\sum_{j=0}^{2^{\ell-2}}Z_j(K_1,\dots,K_\ell)\mu^j,$$

where each Z_j is a $(2^{\ell-2}-j)$-homogeneous polynomial with integer coefficients of ℓ variables and whose coefficients only depends on $(j,\ell,\rho_1,\dots,\rho_\ell)$. Point i), Point ii) and Point iv) are checked.

Let us prove Point iii). We have to explain why $\mu\mapsto W(\mu+K_1,\dots,\mu+K_\ell)$ is a nonzero polynomial for any fixed tuple $(K_1,\dots,K_\ell)\in(0,+\infty)^\ell$ of **distinct** numbers. From the above construction, the formula (55) implies the following one

$$W(\mu+K_1,\dots,\mu+K_\ell)^2=\prod_{(\omega_1,\dots,\omega_\ell)=\{\pm1,\dots,\pm1\}}\left(\sum_{k=1}^{\ell}\omega_k\rho_k\sqrt{\mu+K_k}\right). \tag{57}$$

If the previous product identically vanishes in the range $\mu\in(0,+\infty)$, then at least one of the analytic functions in the product also identically vanishes. This is impossible because the family of functions $(\sqrt{\mu+K})_{K\geq 0}$ is linearly independent with respect to μ.

Remark 25. — *In the case where n is odd, $\ell=n+2$ and $(\rho_1,\dots,\rho_{n+2})=(1,\dots,1)$, there is an easier way to prove Point iii). Indeed, it is sufficient to remark that the coefficient of $\mu^{2^{\ell-1}}$ of $W(\mu+K_1,\dots,\mu+K_\ell)^2$ in the right-hand side of (57) does not vanish. This is obvious because $\sum_{k=1}^{\ell}\omega_k$ is an odd integer for any $\omega_1=\pm1,\dots,\omega_\ell=\pm1$. In other words, (54) gives the inclusion $\mathbf{D}_{2^n}\subset\mathbf{M}_n$ discussed in the introduction.*

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